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Brane boxes and branes on singularities

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Abstract: Brane Box Models of intersecting NS and D5 branes are mapped to D3 branes at C^3/Γ singularities and viceversa, in a setup which gives rise to $N = 1$ supersymmetric gauge theories in four dimensions. The Brane Box Models are constructed on a two-torus. The map is interpreted as T-duality along the two directions of the torus. Some Brane Box Models contain NS fivebranes winding around (p, q) cycles in the torus, and our method provides the geometric T-dual to such objects. An amusing aspect of the mapping is that T-dual configurations are calculated using $D = 4$ $N = 1$ field theory data. The mapping to the singularity picture allows the geometrical interpretation of all the marginal couplings in finite field theories. This identification is further confirmed using the AdS/CFT correspondence for orbifold theories. The AdS massless fields coupling to the marginal operators in the boundary appear as stringy twisted sectors of S^5/Γ. The mapping for theories which are non-finite requires the introduction of fractional D3 branes in the singularity picture.

Keywords: Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories, Supersymmetry and Duality.

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1. Introduction

Brane Boxes are good objects for studying aspects of $N = 1$ supersymmetric chiral gauge theories in four dimensions and their corresponding dimensional reductions.

These objects correspond to a simple generalization of the idea in [1]. In [1], a Dp brane is stretched in between a pair of two NS branes with one direction being finite. The low energy effective field theory which lives on the Dp brane is therefore a $p+1$ dimensional theory compactified on an interval with length given by the distance between the two NS branes. For small enough interval, the field theory on the brane is p dimensional. In addition, the NS branes impose boundary conditions which remove some of the fields which live on the D-brane as well as reduce the supersymmetry by half. The resulting theory is a p dimensional theory with 8 supercharges and p is less or equal to 6. When two brane intervals touch each other there are additional massless multiplets. with 8 supercharges, these correspond to bi-fundamental hypermultiplets which transform under the two gauge groups which leave on the two brane intervals.

Brane boxes [2] are a generalization of this idea to a two dimensional interval. A Dp brane is stretched in between two pairs of two NS branes with two directions being finite. The low energy effective field theory which lives on the Dp brane is therefore a $p+1$ dimensional theory compactified on two intervals with each length given by the distance between the two corresponding NS branes. For small enough interval, the field theory on the brane is $p-1$ dimensional. The NS branes impose boundary conditions which remove more fields which live on the D-brane as well as reduce the supersymmetry by a further half. The resulting theory is a $p-1$ dimensional theory with 4 supercharges and p is less or equal to 5. There are additional states which are massless when two brane boxes touch. With 4 supercharges, these correspond to bi-fundamental chiral multiplets which transform as fundamental and anti-fundamental of the corresponding gauge groups. A superpotential term is present when three such boxes meet. Three chiral multiplets form into a dimension 3 singlet operator which contributes to the superpotential.

Using these rules, a large class of finite chiral $N=1$ supersymmetric gauge theories were constructed in [3]. The beta functions of the gauge theories are directly related to the bending of the branes. Models with vanishing beta function correspond to configurations with no bending. An important class of models was introduced which are not finite but still have some exact marginal operators. Such models obey a "sum of diagonal rule". This rule was independently considered in [4] from a different point of view and is discussed in the next paragraph.

The bending of the branes remains an open question. For weak coupling the NS branes are much heavier than the D branes and are weakly affected by them. For finite string coupling, NS branes which have no balance of D branes ending on them start to bend. It is an important problem to understand such a bending. This information will provide us an understanding of the dynamics of strongly coupled chiral gauge theories. A first step towards understanding such a bending was done in [4]. A constraint on the possible ranks of four gauge groups which meet on a vertex of four brane boxes was derived. This condition requires smoothness of asymptotic bending. In some cases like supersymmetric Yang-Mills theories we need not expect such smoothness and so there is a large class of gauge theories for which this condition is too restrictive. Nevertheless, models which do obey these conditions are nicely behaved models which can teach us a lot on the dynamics of the corresponding gauge theories. As a simple example, brane configurations which satisfy these conditions are anomaly free [4]. Other aspects of the brane boxes and their beta function were considered in [5].

In a different approach, initiated in [6], branes on singularities were analyzed and were shown, in some cases, to have gauge theories which are identical to models derived in the approach discussed above. In this paper we study this correspondence by showing that the two approaches are related by T-duality along two directions. A similar analysis for theories with 8 supercharges appears in [7]. In theories with four supercharges, the constructions of [8, 9] were shown to be equivalent to brane box construction in [2], using T-duality along one direction.

The paper is organized as follows. In section 2 we present two different constructions which lead to four dimensional $N = 1$ supersymmetric gauge theories. We start by reviewing the Brane Box Models of [2]. Then we proceed to review the branes at singularities of [6, 10, 11, 12, 13]. Other related work appears in [14, 15, 16, 17]. We describe the rules which lead to the calculation of the gauge groups, the matter content and the particular cubic interaction terms in the superpotential. One important set of Brane Box models has non-trivial identifications when going around the circle on which they are defined. These models are reviewed in this section to be discussed as the general models in the next sections. The models of branes at singularity which are studied in this paper correspond only to Abelian discrete subgroups of $SU(3)$. Non-Abelian subgroups are harder to map to Brane Box Models and are left for further study.

In section 3, we present a constructive method of building a Brane Box Model from a given singularity model. The construction is formal and serves as a prepartion to a T-duality relation between the two types of setups.

In section 4, we use T-duality to construct a brane at singularity from a given Brane Box Model. This method allows us to calculate T-dual pairs between NS branes which wrap a torus in various ways and a singularity of the form C^3/Γ, with Γ a discrete subgroup of $SU(3)$. This is one of the amusing aspects of the present paper in which four dimensional $N = 1$ supersymmetric gauge theories are used to calculate non-trivial dual pairs in Type II superstring theory.

In section 5 we discuss the counting of marginal couplings for the general class of models introduced in section 2. These parameters receive a geometric interpretation in the brane box picture as distances between NS fivebranes, and Wilson lines around compact directions. In the singularity picture they correspond to integrals of the Type IIB RR and NS two-forms over two-cycles implicit in the singularity. As a check of this identification, we use the AdS/CFT correspondence, and relate the four-dimensional $\mathcal{N} = 1$ finite theories to string theory on $AdS_5 \times S^5/\Gamma$. The massless scalar fields propagating in AdS_5 which couple to the marginal operators in the boundary are seen to arise from stringy twisted sectors of S^5/Γ. This identification of the gauge theory parameters in the brane pictures allow us to make some qualitative statements about the strong coupling regime of the gauge theories.

In section 6 we discuss models which are not conformal field theories. The different gauge theories in the Brane Box models are mapped to fractional branes living on singularities. This leads us to a special class of models constructed from "sewing" three different $N = 2$ models into a general $N = 1$ model. Each $N = 2$ model has a Coulomb branch which becomes part of the Higgs branch of the $N = 1$ model. Using the $N = 2$ beta functions, the one loop beta function for the $N = 1$ model is calculated and is given an interpretation in the brane picture.

2. The constructions

2.1. Overview of the brane box configurations

The $N = 1$ models we will be considering are constructed in a brane setup, in the spirit of [1], which was described in detail in [2]. The description here will be short and further details are contained in [2].

We are working in Type IIB superstring theory with the following set of branes.

- NS branes along 012345 directions
- NS′ branes along 012367 directions
- D5 branes along 012346 direction.

The D5 branes will be finite in two of the directions, 4 and 6; their low-energy effective world volume theory is 3+1 dimensional. The presence of all branes breaks supersymmetry to 1/8 of the original supersymmetry, and thus we are dealing with $N = 1$ supersymmetry (4 supercharges) in four dimensions. The 4 and 6 directions are circles with radii R_4 and R_6 respectively.

A generic configuration consists of a grid of k NS branes and k' NS′ branes in the 46 plane. This divides the 46 plane into a set of kk' boxes. In each box, we can place an arbitrary number of D5 branes. Let $n_{i,j}$ denote the number of D5 branes in the box i, j, $i = 1, \ldots, k$, $j = 1, \ldots, k'$. In the following, indices will denote variables in a periodic fashion: an index i is to be understood modulo k and an index j is to be understood modulo k'. Thus a model's gauge and matter content is specified by the numbers k and k' and the set of numbers $\{n_{i,j}\}$.

The gauge group is $\prod_{i,j} SU(n_{i,j})$. The matter content of the model consists of three types of $N = 1$ chiral representations. They will be denoted as $H_{i,j}$, $V_{i,j}$ and $D_{i,j}$, corresponding to the horizontal, vertical and diagonal multiplets which arise in the brane system (see the details in [2]). $H_{i,j}$ transforms in the $(\Box, \overline{\Box})$ of $SU(n_{i,j}) \times SU(n_{i+1,j})$, $V_{i,j}$ transforms in the $(\Box, \overline{\Box})$ of $SU(n_{i,j}) \times SU(n_{i,j+1})$ and $D_{i,j}$ transforms

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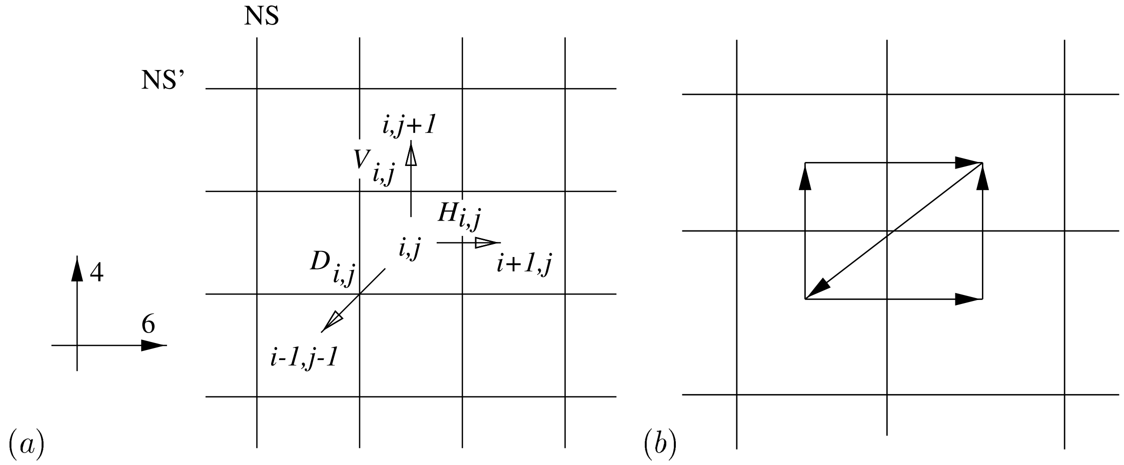

Figure 1: (a) Conventions for denoting the chiral multiplets which are in the fundamental of the group $SU(n_{i,j})$ and in the antifundamental of an adjacent group. (b) The two superpotential couplings at each corner are represented by an oriented triangle of arrows.

in the $(\square, \bar{\square})$ of $SU(n_{i,j}) \times SU(n_{i-1,j-1})$. Figure 1$a$ shows the conventions for denoting the multiplets.

The superpotential in these models is calculated using the rules described in [2]. It is given by

$$W = \sum_{i,j} H_{i,j} V_{i+1,j} D_{i+1,j+1} - \sum_{i,j} H_{i,j+1} V_{i,j} D_{i+1,j+1}. \tag{2.1}$$

The first term corresponds to lower triangles of arrows and the second term corresponds to upper triangles of arrows in the notation of [2], as shown in figure 1b. Note the relative minus sign between the two terms.

2.1.1. Models with non-trivial identifications

The brane box models described above are defined on a torus in the 46 direction in which the NS branes are trivially identified when going around any of the circles. In this section we will review brane box configurations which have non-trivial identifications once going around one of the circles of the torus. The simplest example of this type of models was given in figure 7 of [3].

The construction goes as follows. For any integers k and k' we can form a $k \times k'$ box model as in the models with trivial identification. Without loss of generality, we can assume that along one of the directions of the torus the NS branes are identified trivially. Let us choose it to be the 4 direction. Let p be an integer between 0 and k. We place a $k \times k'$ box model on top of another such box shifted to the left by p boxes. This gives NS$'$ branes which are trivially identified when going around the 4 circle. On the other hand the NS branes are identified non-trivially. These models will be discussed further in subsection 4.2.

$p = 0$ corresponds to the models with trivial identification which are discussed in the previous subsection.

2.2. Marginal couplings

The gauge couplings of the various gauge groups receive contributions from various sources. The simplest contribution is expressed in terms of the positions of the NS branes in the x^6 direction and the position of the NS$'$ branes in the x^4 direction. There are k positions x_6^i and k' positions x_4^j. Correspondingly, the x_6 direction is divided into k intervals with lengths $a_i = x_6^i - x_6^{i-1}$, such that $\sum_i a_i = R_6$. The x_4 direction is divided into k' intervals of length $b_j = x_4^j - x_4^{j-1}$, such that $\sum_j b_j = R_4$. The simplest contribution to the gauge coupling $g_{i,j}$ for the group $SU(n_{i,j})$ is given by

$$\frac{1}{g_{i,j}^2} = \frac{a_i b_j}{g_s l_s^2}. \tag{2.2}$$

The kk' gauge couplings are not all independent. In equation (2.2) they are given by $k + k' - 1$ parameters corresponding to the positions of the NS and NS$'$ branes. Two positions can be set to zero by the choice of origin in the 46 directions, but the area of the 46 torus gives one more parameter. The couplings do not depend on the ratio between the two radii of the torus. As we will see later, using the mapping to the branes on singularities, the field theories often have more than $k + k' - 1$ parameters, indicating that we have not identified all of the contributions to these couplings.

The theta angles of the gauge theories receive various contributions. Let A_i be the gauge field on the world volume of the i^{th} NS brane and A'_j be the gauge field on the world volume of the j^{th} NS$'$ brane. Since the dimensions 4 and 6 are compact, there can be non-zero Wilson lines of A_i along 4, and of A'_j along 6. Let $R_{i,j}$ denote the area in the 46 direction which is bounded by the pair of NS branes and NS$'$ branes. The theta angle for the i, j group depends on the line integral of the different gauge fields along the boundary of $R_{i,j}$. Schematically,

$$\theta_{i,j} = \int_{R_{i,j}} B + \int_{a_i} (A'_{j-1} - A'_j) + \int_{b_j} (A_i - A_{i-1}). \tag{2.3}$$

where B is the RR two form of Type IIB superstring theory. The contributions from the gauge fields are required for the invariance of $\theta_{i,j}$ under gauge transformations of B. Were this the entire story we would again have $k + k' - 1$ parameters. However, invariance under gauge transformations of the one-forms require that additional terms be added to this expression involving axion-like fields living at the intersections of the NS and NS$'$ branes.

In general, when quantum effects are taken into account these quatities run accordingly with the renormalization group. However, the brane configurations allow for a simple construction of $N = 1$ theories which have some marginal couplings, *i.e.* a submanifold of renormalization group fixed points in the space of couplings. For example, the field theory analysis in [3] showed that models which satisfy the "sum of diagonals rule"

$$n_{i,j} + n_{i+1,j+1} = n_{i+1,j} + n_{i,j+1}\,, \tag{2.4}$$

give rise to non-finite models which have some marginal operators. This condition was discussed in the context of brane bending in [4]. We will discuss issues related to this condition in section 6. The simplest models verifying these conditions are those in which all $n_{i,j}$ are equal. They give rise to exactly finite theories, in which no parameter is renormalized.

Let us now discuss the number of independent parameters which contribute to the gauge couplings. We claim that there are actually

$$k + k' + r - 2$$

such couplings, where r is the greatest common divisor of k and k', $r = \gcd(k, k')$. For models with non-trivial identifications (of the type described in paragraph 2.1.1) the number of marginal couplings generalizes to

$$k + \gcd(k, p) + \gcd(k, k' + p) - 2\,.$$

This counting follows from the field theory analysis performed in [3]. From the point of view of the brane box construction, we will give few arguments which support this claim. Further evidence will be provided in section 5, using the T duality with the picture of branes at singularities.

First consider from a field theory point of view the asymmetry in the construction in terms of branes. There are matter fields which come from horizontal, vertical and diagonal arrows, however, the parameters which are seen in equation (2.2) correspond only to horizontal and vertical distances. There are no parameters which correspond to diagonal fields. To make the discussion more clear, let us center on a $k \times k'$ box model with trivial identifications, and let us define the following quantities.

$$h_i = \sum_{j=1}^{k'} \frac{1}{g_{i,j}^2}, \qquad v_j = \sum_{i=1}^{k} \frac{1}{g_{i,j}^2}, \qquad d_l = \sum_{diagonals} \frac{1}{g_{i,j}^2}\,. \tag{2.5}$$

The sum over diagonals means that one starts with some box and then proceeds along the diagonal arrows until coming back to the first box. There are r different such diagonals which correspond to r different parameters, d_l and each sum contains kk'/r summands such that each gauge coupling appears exactly once in one of the parameters d_l. We claim that these quantities form a set of independent paramters which give rise to the various gauge couplings.

The asymmetry becomes more clear when we consider a different brane box construction which gives rise to the same field theory. For a given field theory, as above, we can always rename what we call the fields H, V and D by any permutation, say D, V and H. There are actually 6 such choices of renaming the fields. While this is a formal process from a field theory point of view which does not change the matter content, it is a crucial difference from the point of view of brane box construction. In the example of permutation chosen, fields which in the original construction come

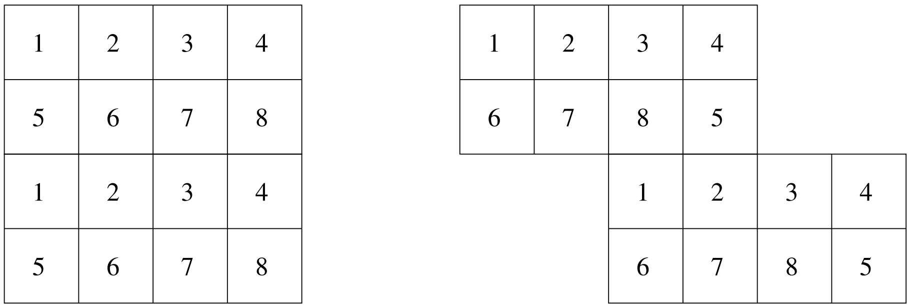

Figure 2: A permutation on the types of fields. In this particular example the H, V, D fields are transformed into H, D, V fields, respectively. The 4×2 box model with trivial identifications is mapped to the 4×2 model with a nontrivial identification with a horizontal jump by $p = 2$. The numbers in the boxes indicate labels of the boxes.

from horizontal arrows, come in the permuted construction from diagonal arrows and viceversa. Vertical fields on the other hand remain vertical.

The example in figure 2 shows how to construct such a brane box permutation. A systematic way of constructing the permuted box model from the original one is as follows. We pick a chain of periodic horizontal boxes. We order the boxes in a new order which is specified by the permutation. From each box of the original horizontal chain we pick a chain of periodic vertical boxes. We put these boxes in the new direction specified by the permutation. The permutation on diagonal chain of boxes is already build into the new setup. The new box model is done.

Let us count parameters in the original model and its corresponding permuted model. In figure $2a$, there are 4 vertical lines and there are 2 horizontal lines. (Here we mean in a unit box of the torus). In addition there are 2 independent closed chains of diagonals (given by 4,7,2,5 and 8,3,6,1). The number of diagonals is given for general k and k' by $r = \gcd(k, k')$. In the permuted model of figure $2b$, there are 2 vertical lines and 2 horizontal lines. On the other hand, there are 4 diagonal chains. In both cases, we will count $4 + 2 + 2 - 2 = 6$ marginal parameters for the field theory. We see, as expected from the permutation, that the number of vertical and the number of diagonal parameters are exchanged, while the number of horizontal lines is not changed.

In any of the models, the diagonal parameters are not easily visible but the vertical parameters are visible. A symmetry of the construction from a field theory point of view, thus implies that indeed we have counted correctly the set of the marginal parameters of the model. This counting will be useful for the identification of parameters in the singularity picture in section 5.

2.3. Overview of the branes at singularity

2.3.1. The spectrum

The dynamics of four dimensional $\mathcal{N} = 4$ gauge theories can be studied by realizing them in the world-volume of parallel Type IIB D3-branes. Let us state, for concreteness, that such world-volume spans the coordinates 0123. In this context many properties of the gauge theory are usefully related to properties of string theory and the brane dynamics. For instance, the $SU(4)_R$ R-symmetry appears as the $SU(4) \approx SO(6)$ rotation group on the transversal coordinates 456789. The gauge coupling constant is given by the string coupling, and Montonen-Olive duality is realized as the ten-dimensional $SL(2,Z)$ of Type IIB superstring theory.

In [13] this idea was extended by introducing a family of four-dimensional gauge field theories with reduced supersymmetry, which can be realized in the world-volume theory of D3 branes sitting at a singular point. This singularities are locally described as C^3/Γ. Here C^3 corresponds to the coordinates transverse to the D3 brane, namely 456789, and Γ is a discrete subgroup of $SO(6) \approx SU(4)$. Since Γ acts on the R-symmetry of the theory, the amount of surviving supersymmetry depends on this action. Theories with $\mathcal{N}=2$ supersymmetry are obtained when $\Gamma \subset SU(2)$, $\mathcal{N}=1$ (generically chiral) gauge theories appear if $\Gamma \subset SU(3)$, and non-supersymmetric theories correspond to Γ being a generic subgroup of $SU(4)$.

The spectrum of the resulting theory can be analyzed using the techniques developed in [6]. We review the result for Γ an *Abelian* subgroup of $SU(3)$, since we will be interested in this particular family of $\mathcal{N}=1$ theories.

Let $|\Gamma|$ denote the order of Γ. A configuration of N D3 branes at the orbifold can be studied on the covering flat space by considering N groups of $|\Gamma|$ D3 branes. Γ acts on the set of $N|\Gamma|$ Chan-Paton factors as N copies of the regular ($|\Gamma|$-dimensional) representation $\mathcal{R}_\Gamma$, *i.e.* $\mathcal{R}_{C.P.} = N\mathcal{R}_\Gamma$ (Other embeddings of the Chan-Paton factors and their interpretation will be discussed in Section 6). When Γ is Abelian it has $|\Gamma|$ unitary irreducible representations $\mathcal{R}_I$, all of which are one-dimensional. The regular representation is reducible and decomposes as $\mathcal{R}_\Gamma = \bigoplus_I \mathcal{R}_I$. One must also define how Γ acts on C^3 to form the quotient singularity; this is specified by a (faithful) three-dimensional representation, which has a decomposition in irreducible representations as:

$$\mathbf{3} \; = \; \mathcal{R}_{A_1} \oplus \mathcal{R}_{A_2} \oplus \mathcal{R}_{A_3} \, . \tag{2.6}$$

In order for $\mathbf{3}$ to be a representation of $SU(3)$ rather than of $U(3)$ there is a requirement on the choice of the representations $\mathcal{R}_{A_i}$, whose statement is easier by noticing the following fact. The set of irreducible representations forms an Abelian group (isomorphic to Γ) with respect to the tensor product of representations. We write the group law as

$$\mathcal{R}_I \otimes \mathcal{R}_J \; \equiv \; \mathcal{R}_{I \oplus J} \, . \tag{2.7}$$

Using this additive notation on the indices of the irreducible representations, the trivial representation is denoted $\mathcal{R}_0$, and $\mathcal{R}_{-I}$ denotes the inverse element of $\mathcal{R}_I$. The requirement on the representation $\mathbf{3}$ can be stated as $\mathcal{R}_{A_1} \otimes \mathcal{R}_{A_2} \otimes \mathcal{R}_{A_3} = \mathcal{R}_0$, or equivalently as $\mathcal{R}_{A_3} = \mathcal{R}_{-A_1-A_2}$.

The construction in [13] results in the following spectrum. The gauge groups[1] contains a factor $SU(N)$ per irreducible representation of Γ, so the group is $\prod_I SU(N)_I = SU(N)^{|\Gamma|}$. The chiral matter is found by computing the products

$$\mathbf{3} \otimes \mathcal{R}_I = \mathcal{R}_{I\oplus A_1} \oplus \mathcal{R}_{I\oplus A_2} \oplus \mathcal{R}_{I-A_1-A_2}\,. \tag{2.8}$$

There are three kinds of chiral multiplets, which are associated to the three complex planes transverse to the D3 branes. We will denote them by $\Phi_{I,I\oplus A_i}$, for $i=1,2,3$. The fields $\Phi_{I,I\oplus A_1}$ transform in the $(\Box,\overline{\Box})$ of $SU(N)_I \times SU(N)_{I\oplus A_1}$, $\Phi_{I,I\oplus A_2}$ transform in the $(\Box,\overline{\Box})$ of $SU(N)_I \times SU(N)_{I\oplus A_2}$, and $\Phi_{I,I\oplus A_3}$ transforms in the $(\Box,\overline{\Box})$ of $SU(N)_I \times SU(N)_{I\oplus A_3}$. Notice there are three such fields per irreducible representation of Γ.

Finally, for each I there are two cubic terms in the superpotential, which takes the form

$$W=\sum_I \left[\Phi_{I,I\oplus A_1}\,\Phi_{I\oplus A_1,I\oplus A_1\oplus A_2}\,\Phi_{I\oplus A_1\oplus A_2,I} \;-\; \Phi_{I,I\oplus A_2}\,\Phi_{I\oplus A_2,I\oplus A_2\oplus A_1}\,\Phi_{I\oplus A_2\oplus A_1,I}\right]. \tag{2.9}$$

Before going further in the discussion of these models and their relation to the brane box configurations, it will be useful to discuss a few examples.

2.3.2. Examples

In the following we describe the spectrum for D3 branes on some singularities. Since the case of $\mathcal{N}=2$ theories (corresponding to an A_k ALE singularity), has been largely discussed in the literature [6], we will mention only $\mathcal{N}=1$ models.

i) C^3/Z_3

Consider a C^3/Z_3 singularity, where the generator θ of Z_3 acts on C^3 as

$$(z_1,z_2,z_3) \to (e^{2\pi i/3}z_1, e^{2\pi i/3}z_2, e^{2\pi i/3}z_3)\,. \tag{2.10}$$

This is the only choice of the representation $\mathbf{3}$ that leaves unbroken $\mathcal{N}=1$ supersymmetry (and not $\mathcal{N}=2$). The group $\Gamma = Z_3$ has three one-dimensional irreducible representations $\mathcal{R}_I$, $I=0,1,2$. The representation $\mathcal{R}_I$ associates to θ the phase $e^{2\pi i I/3}$. Clearly the product law in the set of irreducible representations is $\mathcal{R}_I \otimes \mathcal{R}_J = \mathcal{R}_{I+J}$, i.e. amounts to usual addition (modulo 3) of subindices. We see from (2.10) that the representation $\mathbf{3}$ that we have chosen decomposes as $\mathcal{R}_1 \oplus \mathcal{R}_1 \oplus \mathcal{R}_1$.

Following the rules above, the gauge group is $SU(N)_0 \times SU(N)_1 \times SU(N)_2$. The fields of type $\Phi_{I,I+A_1}$, associated to the first complex plane, transform as a copy of $(\Box_0,\overline{\Box}_1)+(\Box_1,\overline{\Box}_2)+(\Box_2,\overline{\Box}_3)$. We will denote these fields as Q_I^1. The fields $\Phi_{I,I+A_2}$, which we denote by Q_I^2, are associated to the second complex plane, and transform as another copy of the same representation. Finally, the fields $\Phi_{I,I+A_3}$ transform again in this

[1] The $U(1)$ gauge fields (but one) are not expected to appear in the low energy dynamics of the D-branes. A possibility is that they are broken by a Green-Schwarz mechanism [18], as happens in certain six dimensional models [6, 19, 20].

representation. We denote them by Q_I^3. So in total the model has nine chiral multiplets Q_I^a transforming in the representation

$$3\,(\Box_0,\overline{\Box}_1)\;+\;3\,(\Box_1,\overline{\Box}_2)\;+\;3\,(\Box_2,\overline{\Box}_3)\,. \tag{2.11}$$

Following the rules above, there is a superpotential which can be rewritten as

$$W \sim \epsilon^{IJK} Q_I^1 Q_J^2 Q_K^3\,. \tag{2.12}$$

This model was studied in [11, 12] before the general formulation of [13] appeared.

ii) $C^3/(Z_k \times Z_{k'})$

Let us consider a rather large family of singularities of type $C^3/(Z_k \times Z_{k'})$, which will be useful in the following sections. Let θ, ω denote the generators of the Z_k, $Z_{k'}$ subgroups, respectively. The group $\Gamma = Z_k \times Z_{k'}$ has kk' irreducible representations, denoted $\mathcal{R}_{a,b}$ ($a = 0,\ldots,k-1$, $b = 0,\ldots,k'-1$). The representation $\mathcal{R}_{a,b}$ associates to the general element $\theta^m\omega^n$ the phase factor $e^{2\pi i\frac{am}{k}} e^{2\pi i\frac{bn}{k'}}$. Notice that *one* uppercase index in the general formulation represents *two* lowercase indices in this case, since the group has two generators. The product of representations is given by $\mathcal{R}_{a,b}\otimes\mathcal{R}_{a',b'} = \mathcal{R}_{a+a',b+b'}$, *i.e.* separate addition in the indices.

Let us choose the action of Γ on C^3 such that the generators act as

$$\begin{aligned} \theta\ &: (z_1,z_2,z_3) \to (e^{2\pi i/k} z_1, z_2, e^{-2\pi i/k} z_3) \\ \omega\ &: (z_1,z_2,z_3) \to (z_1, e^{2\pi i/k'} z_2, e^{-2\pi i/k'} z_3)\,. \end{aligned} \tag{2.13}$$

This means that the corresponding representation $\mathbf{3}$ decomposes as $\mathbf{3} = \mathcal{R}_{1,0} \oplus \mathcal{R}_{0,1} \oplus \mathcal{R}_{-1,-1}$.

The gauge group is $\prod_{a,b} SU(N)_{a,b} \approx SU(N)^{kk'}$. The chiral multiplets associated to the first complex plane, $\Phi_{I,I+A_1}$, form the representation $\bigoplus_{a,b}(N_{a,b},\overline{N}_{a+1,b})$. Similarly, the fields $\Phi_{I,I+A_2}$ are in the representation $\bigoplus_{a,b}(N_{a,b},\overline{N}_{a,b+1})$, and the fields $\Phi_{I,I+A_3}$ transform as $\bigoplus_{a,b}(N_{a,b},\overline{N}_{a-1,b-1})$.

The resulting spectra are rather lengthy to list, but straightforward to obtain. Similarly, the superpotential terms follow from equation (2.9).

We postpone the discussion of how the field theory parameters are encoded in the configuration of branes at singularities until Section 5. In the meantime, in sections **??** and **??**, we establish a connection between the brane box models and the branes at singularities. This relation will allow us to map several parameters, states, and field theory phenomena from the brane box configurations to the singularity language.

3. From the singularity to the brane box

3.1. The construction

The general pattern of the theories that we have obtained studying D3 branes on top of C^3/Γ singularities, for Abelian Γ, is very reminiscent of the type of theories we obtained using the brane box configurations. Our aim in this section is to show that actually for each such singularity theory one can construct a suitable brane box configuration leading to the same four-dimensional $\mathcal{N} = 1$ gauge theory. We stress that the argument is simply based on the matching of the spectra, and does not establish a physical principle underlying the correspondence. We will come back to this point in the following section, where we show the relation follows from T duality.

The general strategy to construct such a brane box configuration is to draw one box for each irreducible representation of Γ, so as to ensure the gauge group is the same, and adjoin these boxes such that the chiral multiplets H, V, D in the box model reproduce the fields $\Phi_{I,I\oplus A_1}$, $\Phi_{I,I\oplus A_2}$ and $\Phi_{I,I\oplus A_3}$. Notice that in the brane box diagram the relation of neighbourhood of boxes will thus be defined in terms of the product law of irreducible representations in Γ. The construction of the brane boxes is thus very similar to that of the quiver diagrams described in [21].

The construction is easily organized as follows: The first step is to draw a row of boxes corresponding to the irreducible representation $\mathcal{R}_0$, $\mathcal{R}_{A_1}$, $\mathcal{R}_{2A_1}$, ..., $\mathcal{R}_{(n_1-1)A_1}$, where n_1 is the order of $\mathcal{R}_{A_1}$ in the set of irreducible representations of Γ. The fact that $\mathcal{R}_{n_1 A_1} \equiv \mathcal{R}_0$ means that the row is compactified on a circle. Equivalently, one can think of a one-dimensional infinite periodic array of boxes, of which the finite set described above is a fundamental region. This construction ensures that the horizontal fields between the boxes reproduce some of the fields $\Phi_{I,I+A_1}$. A picture of this first step in the construction is shown in figure 3.

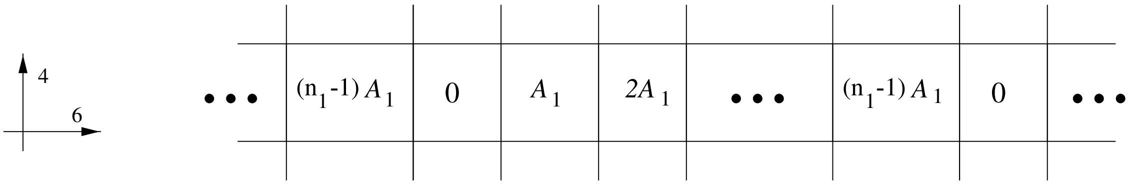

Figure 3: First step in the construction of the brane box model corresponding to a singularity C^3/Γ. The labels on each box denote the irreducible representation of Γ associated to the gauge factor in that box. The basic period in the infinite array of boxes is bounded by dark lines.

Next, from each of the boxes $\mathcal{R}_{nA_1}$ in the row, we start a vertical column of boxes, which we label $\mathcal{R}_{nA_1}$, $\mathcal{R}_{nA_1\oplus A_2}$, ..., $\mathcal{R}_{nA_1\oplus(n_2-1)A_2}$, where n_2 is the order of $\mathcal{R}_{A_2}$ in the set of irreducible representations. Again, since $\mathcal{R}_{n_2 A_2} \equiv \mathcal{R}_0$, there is an identification of the horizontal sides of the resulting rectangle, which makes the configuration to be compactified on a two-torus. Alternatively, one can extend the construction to the full plane, and think of it as the universal cover of the torus. This construction ensures that the new horizontal arrows reproduce the fields of type $\Phi_{I,I\oplus A_1}$, and the vertical arrows the multiplets of type $\Phi_{I,I\oplus A_2}$. Quite remarkably, the fields of type $\Phi_{I,I+A_3}$ are automatically reproduced by the diagonal arrows, and the superpotential interactions (2.9) are reproduced by the triangle rule (2.1). In figure 4 we show a typical region in

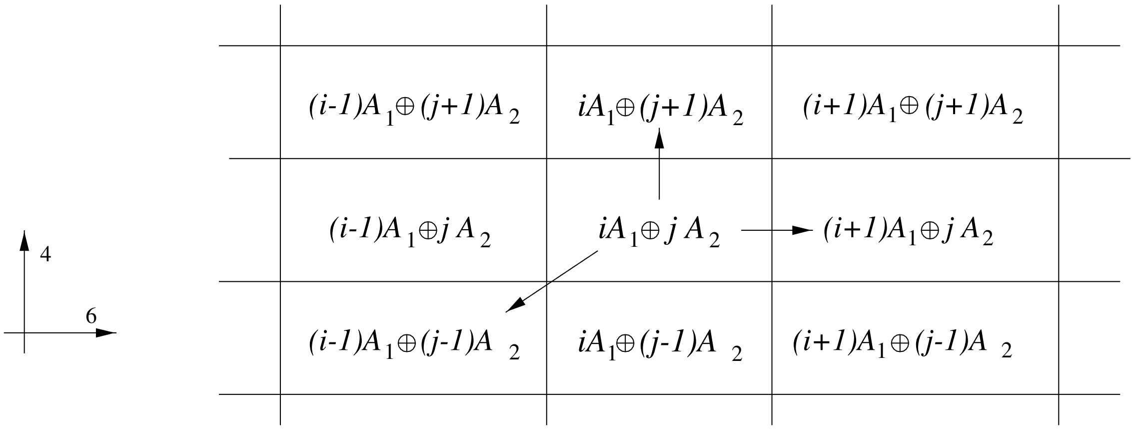

Figure 4: A typical region in a brane box model constructed from a singularity. The arrows denote the chiral multiplets $\Phi_{I,I\oplus A_1}$, $\Phi_{I,I\oplus A_2}$, and $\Phi_{I,I-A_1-A_2}$, for $I = iA_1 \oplus jA_2$, which appear as the horizontal, vertical and diagonal fields.

a general brane box thus constructed. As can be read in the picture, when one moves horizontally to the right, the label in the boxes shifts by A_1; when one moves vertically upwards, the label changes by A_2; finally, a diagonal movement from upper-right to lower-left shifts the label by $-A_1 - A_2$.

An important question is whether all irreducible representations of Γ indeed appear in this rectangle. That this is so follows from the fact that the representation **3** was chosen to be faithful. This means that all irreducible representations of Γ are generated by $\mathcal{R}_{A_1}$, $\mathcal{R}_{A_2}$. As an example of a non-faithful representation, consider the case where $\Gamma = Z_8$ and the representation **3** decomposes as $\mathcal{R}_2 \oplus \mathcal{R}_2 \oplus \mathcal{R}_4$. It is easy to construct the brane configuration and to discover that it actually describes the case with $\Gamma = Z_4$ and the representation **3** decomposes as $\mathcal{R}_1 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2$.

Another related point is whether all the $n_1 n_2$ boxes are actually different. In general, this is not so, and each box will be repeated $q = n_1 n_2/|\Gamma|$ times. Since this $n_1 \times n_2$ box region is the *smallest* rectangle with *trivial* identifications of sides, and the true unit cell (where each box appears once and only once) is *smaller*, it will have *non-trivial* identification of sides. The true unit cell can be obtained as a $n_1 \times |\Gamma|/n_1$ cell in the rectangle. The relation between the $n_1 \times n_2$ rectangle and the unit cell is illustrated in figure 5. It is clear that the identification of vertical sides of the unit cell will be trivial. However, the identifications of horizontal sides is accompanied by a shift. If the box marked with a 'x' in the picture is labeled $\mathcal{R}_0$, the box marked with a '*' is labeled $\mathcal{R}_{(|\Gamma|/n_1)A_2}$, which is equal to $\mathcal{R}_{pA_1}$ for some integer p. The identification of horizontal sides is shifted by p boxes to the left.

As an illustrative case consider the orbifold C^3/Z_3, discussed as example i) in section 2.2, where we had $\mathbf{3} = \mathcal{R}_1 \oplus \mathcal{R}_1 \oplus \mathcal{R}_1$. Since the order of $\mathcal{R}_1$ in the set of irreducible representations is 3, we have $n_1 = 3$ and also $n_2 = 3$. The procedure we have described yields a 3×3 rectangle with trivial identifications, which is depicted in figure 6. Each box is repeated 3 times ($q = 3$), so the true unit cell is smaller, and has non-trivial

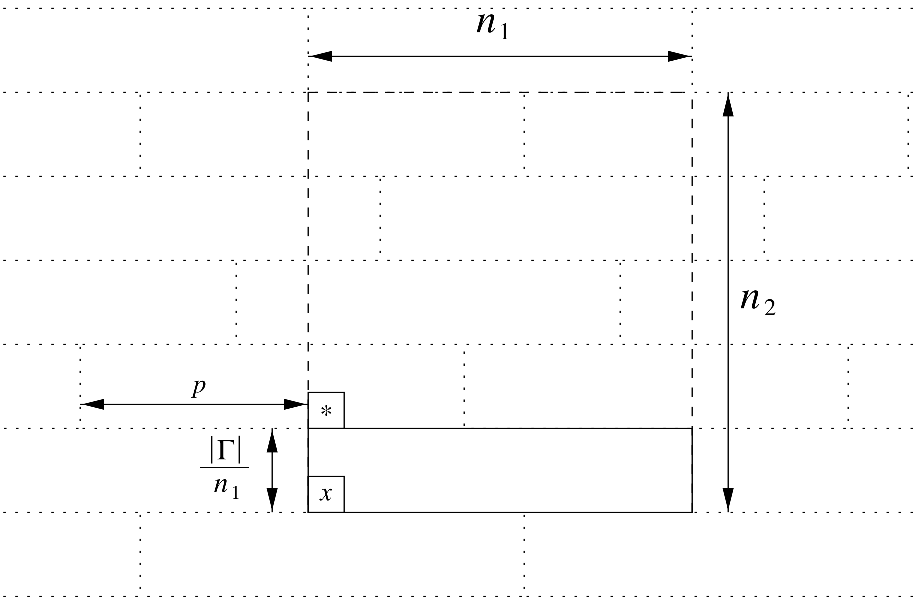

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Figure 5: The relation between the $n_1 \times n_2$ box rectangle and the true unit cell in a general brane box configuration.

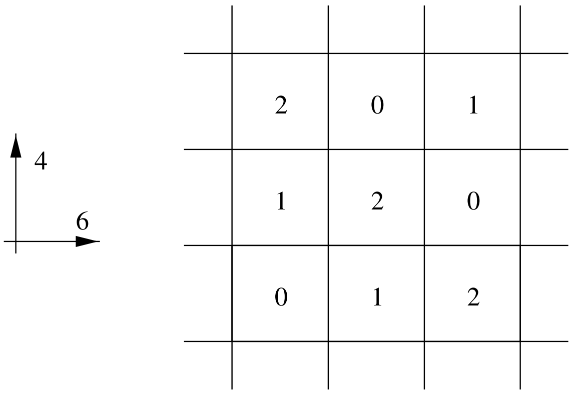

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Figure 6: The brane box corresponding to a C^3/Z_3 singularity. It is a 3×1 box model with trivial identifications of vertical sides, and identification of horizontal sides up to a shift of one box.

identifications. The unit cell can be taken to be the 3×1 cell highlighted in the figure. The vertical sides are identified in the trivial way, but the horizontal identification is accompanied by a shift of one box ($p = 1$). This brane configuration was introduced in [3], where it was already observed that its spectrum matched that of D3 branes at a Z_3 singularity.

An important point is that the consistency of both constructions is only possible for *Abelian* discrete groups Γ. This is suggested from a number of perspectives. For example, if one wishes to construct *finite* theories from D3-branes sitting at singularities, one should choose the Chan-Paton embedding as N copies of $\mathcal{R}_\Gamma$. The gauge group for a general Γ is $\prod_I SU(Nn_I)$, where n_I is the dimension of the I^{th} irreducible representation. In the construction of *finite* theories using brane box configurations [3], the gauge group is $SU(N)^M$, where M is the number of boxes. Thus the brane box configurations reproduce some of the finite models that can be constructed from singularities, namely those where Γ is Abelian and all the irreducible representations are one-dimensional, $n_I = 1$, $\forall I$.

Also, when Γ is Abelian the tensor product of representations is commutative, so $\mathcal{R}_{I\oplus A_1\oplus A_2} \equiv \mathcal{R}_{I\oplus A_2\oplus A_1}$. This is a necessary requirement in our construction of the brane boxes, since it ensures that, starting from the box labeled by $\mathcal{R}_I$ and moving one box to the right and then one box upwards one reaches the same box than moving first upwards and then to the right, an unavoidable geometrical fact in the brane box construction.

This restriction on the type of singularity is hardly a surprise. Several works in geometric engineering (see e.g. [22]) have shown that the geometric approach to the realization of gauge theories is more *general* than the constructions using brane configurations. However we would like to stress that in many instances the brane configurations provide a *simpler* realization of the gauge theories. In our particular case, we are to see that configurations which are *not* finite are easily constructed and analyzed in the brane configuration language, by simply putting a different number of D5 branes in each box. Reproducing these theories using branes at singularities requires the use of fractional branes, objects which are forced to lie at the singular point. Since basically everything is happening at the singular point, the construction is much less intuitive. Somehow, the brane picture 'opens up' all the phenomena happening at the singularity and displays them in different boxes.

As a final comment, let us mention that there is an arbitrary choice in the procedure above, namely the different ways to assign the three kinds of fields H, V, D to the three kinds of fields $\Phi_{I,I+A_i}$ $i = 1,2,3$. There are six such inequivalent choices. The brane box configurations one obtains are related in such a way that, e.g. the fields that arise from horizontal arrows in one come from diagonal arrows in the other. This is clearly the transformation of brane box models we introduced in section 2. The meaning of this transformation will become clear after the T-duality relation between branes boxes and branes at singularities is established in the next section.

3.2. Examples

To illustrate the ideas we have introduced, we present a few examples of the construction above.

i) Consider a singularity C^3/Z_k, with the generator θ of Z_k acting on C^3 as

$$\theta \;:\; (z_1, z_2, z_3) \to (e^{2\pi i/k} z_1, e^{2\pi i \frac{p}{k}} z_2, e^{2\pi i \frac{(-p-1)}{k}} z_3)\,, \tag{3.1}$$

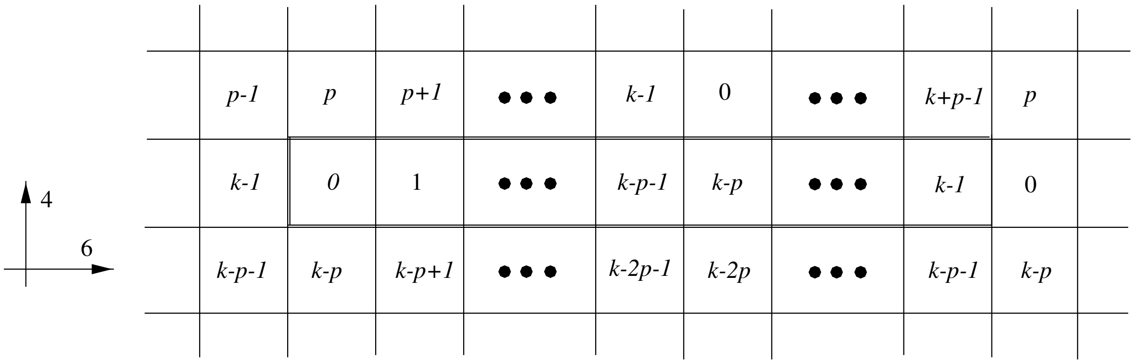

Figure 7: The brane box configuration obtained form the C^3/Z_k singularity described in the text. It corresponds to a $k \times 1$ box model with trivial identifications of vertical sides and identifications of the horizontal sides up to a shift of p boxes to the left.

with p an integer in the range $0 \leq p \leq k-1$. The representation **3** we have chosen decomposes as $\mathcal{R}_1 \oplus \mathcal{R}_p \oplus \mathcal{R}_{-p-1}$. It is easy to check that the above procedure yields the brane box diagram shown in figure 7. Starting from any box, a horizontal movement to the right shifts the label by 1, so that the horizontal arrows give rise to the fields $\Phi_{I,I+1}$. A vertical movement upwards shifts the label by p, so that vertical arrows correspond to $\Phi_{I,I+p}$. And a diagonal movement downwards and to the left shifts the label by $-p-1$, so that diagonal fields correspond to $\Phi_{I,I-p-1}$.

The order of $\mathcal{R}_{A_1}$ is k. For $p \neq 0$ the order of $\mathcal{R}_{A_2}$ is k/ℓ, where ℓ is the greatest common divisor of k and p, $\ell = \gcd(k,p)$. Our procedure above yields a rectangle of $k \times k/\ell$ boxes with trivial identifications. Each box is repeated k/ℓ times, so the unit cell is smaller and has non-trivial identification of sides. In figure 7 we show a choice of unit cell, which has trivial identifications of vertical sides and the horizontal identification up to a shift of p boxes to the left. For $k=3$, $p=1$, we recover the Z_3 example previously studied.

ii) As a last example consider the singularities of type $C^3/(Z_k \times Z_{k'})$, with the action on C^3 defined as in (2.13): $\mathbf{3} = \mathcal{R}_{1,0} \bigoplus \mathcal{R}_{0,1} \bigoplus \mathcal{R}_{-1,-1}$. It is easy to realize that the spectra of the gauge theories on D3 branes on these singularities can be obtained by D5 branes on a grid of $k \times k'$ boxes, with trivial identifications of sides. One such grid is shown in figure 8, where each box is labeled by its associated irreducible representation. Horizontal movements to the right change the label by multiplication by $\mathcal{R}_{1,0}$, vertical movements upwards correspond to multiplication by $\mathcal{R}_{0,1}$, and diagonal movements to multiplication by $\mathcal{R}_{-1,-1}$. Since the order n_1 of $\mathcal{R}_{1,0}$ in the set of irreducible representations is k, and the order n_2 or $\mathcal{R}_{0,1}$ is k' in this case the $n_1 \times n_2$ rectangle coincides with the unit cell.

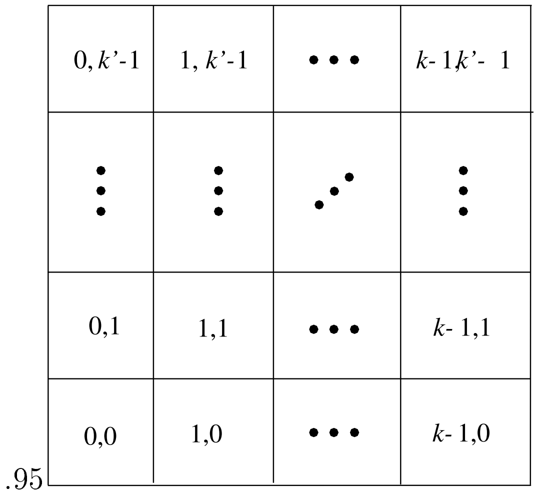

Figure 8: The unit cell of a $k \times k'$ box model with trivial identifications, as obtained from a $C^3/(Z_k \times Z_{k'})$ singularity. The cell extends in plane 4-6.

4. T-duality: from the brane box to the singularity

4.1. Some simple examples

In this section we will explain the reason underlying the precise matching found above between the spectra of brane box configurations and D3 branes on singularities. Specifically, we argue that the relation is a T-duality between both kinds of constructions. We will show how, starting from a brane box configuration and T-dualizing along 4 and 6, the geometry around the D3 branes in the dual is locally C^3/Γ, with Γ an abelian subgroup of $SU(3)$. We also provide an explicit construction of the Γ corresponding to a given brane box configuration.

The D5 branes will not play any relevant role in the T-duality relation between the NS and NS' branes and the singularity. For simplicity it is convenient to consider the case in which the number of D5 branes in each box is the same. More general configurations will be analyzed in Section 6

The T-duality transformation is quite analogous to that relating a set of k parallel NS branes and an A_{k-1} ALE space, so it is convenient to briefly review some of its features. Consider a Type IIB configuration of k parallel NS branes extending along 012345, and let x^4, x^6 be compact coordinates. If N D5 branes are located along 012346 (wrapping the two-torus in 46) the configuration provides a realization of the $SU(N)^k$ $\mathcal{N}=2$ elliptic models [23] on the D brane worldvolume. The T-duality considerations in this case have already been explored in [7].

Performing a T-duality along 4,6, the D5 branes are mapped to D3 branes along 0123, sitting at a point in the T-dual coordinates, which we denote by $4'$, $6'$. The duality along 4 is longitudinal to the k NS branes, and does not change them, while the duality along 6 transforms them into k Kaluza-Klein (KK) monopoles. Thus, the space parametrized by $6'789$ in the T-dual is a k-centered multi-Taub-NUT space, described by the metric

$$
\begin{aligned}
ds^2 &= \frac{V}{4} d\vec{r}^{\,2} + \frac{V^{-1}}{4}(dx^{6'} + \vec{\omega}\cdot d\vec{r})^2, \\
\text{with} \quad V &= 1 + \sum_{i=1}^{k} \frac{1}{|\vec{r}-\vec{x}_i|}
\end{aligned}
\tag{4.1}
$$

and $\vec{\nabla}\times\omega = \vec{\nabla} V$. This is a fibration of an S^1 (parametrized by $x^{6'}$) over R^3 (parametrized by $\vec{r} = (x^7, x^8, x^9)$), the fibers of which shrink to zero radius at the k centers $\vec{x}_i$. The parameters in the original theory can be traced to the final configuration. For example, the position of the k NS branes on 789 are mapped to the positions of the k centers $\vec{x}_i$. An interesting remark in this respect is that, when all such positions coincide in the brane box configuration, all the centers in the Taub-NUT space coalesce at a point. For $\vec{r}$ very close to this point, the constant term in V in equation (4.1) can be neglected, and the geometry is that of an ALE singularity. If in the initial picture the D5 branes also sit at $x^7 = x^8 = x^9 = 0$, the D3 branes will be located at the singular point, and the physics of the gauge theory on their worldvolume is controlled by the structure of the A_{k-1} singularity. This provides the connection with the description in [6, 13]. When the positions of the centers differ from each other, the singularity is resolved and the number of factors in the gauge theory is reduced, by Higgs breaking to diagonal subgroups.[2] This is the same breaking that occurs in the initial brane box configuration when the positions of the NS branes are slightly changed.

It may seem that the positions of the NS branes on x^6 have been lost in the T-duality. However, as shown for instance in [24], they are actually encoded in the singularity picture as integrals of the NS-NS two-form B_{NS} over the non-trivial 2-cycles of the Taub-NUT (or ALE) space. Such two-spheres, which we will denote by Σ_{ij} are obtained as the fibration of the S^1 parametrized by $x^{6'}$ over the segments joining the centers $\vec{x}_i$ and $\vec{x}_j$. A basis of $k-1$ two-cycles is provided by $\Sigma_{i,i+1}$, for $i = 1, \ldots, k-1$. So, the $k-1$ independent quantities

$$
a_i = \int_{\Sigma_{i,i+1}} B_{NS}\,. \tag{4.2}
$$

provide the $k-1$ independent distances between NS branes (more precisely, they provide the ratios of such distances to the total length R_6 of the x^6 direction). There is also a set of analogous parameters corresponding to

$$
v_i = \int_{\Sigma_{i,i+1}} B_{RR}\,, \tag{4.3}
$$

where one is integrating the Ramond-Ramond two-form field over the basic two-cycles. As discussed in [3], these correspond to the differences of Wilson lines along x^4 of the

[2]Notice that here we are considering blow-ups of small size as compared with the Taub-NUT radius, so that the approximation of the metric as an ALE space remains valid.

world-volume $U(1)$ gauge fields of the original k NS branes (more precisely, the ratios of such differences to dual radius $1/R_4$). The parameters a_i, v_i define the gauge couplings of the $\mathcal{N}=2$ gauge theory [13, 3], as also follows by particularizing equations (2.2), (2.3) to this $\mathcal{N}=2$ case.

The Coulomb branch of the gauge theory in the singularity language is parametrized by movements of the D3 branes in $4'5$ keeping the coordinates in $6'789$ fixed at the singularity. As we have mentioned, there are Higgs branches which correspond to resolving partially the singularity (these map to the removal of the corresponding NS brane in the brane box picture). Finally, there is also a Higgs branch corresponding to moving the D3 branes away from the singularity (this branch maps to recombining the D5 branes and moving them away from the grid of NS and NS' branes).

Thus we see how all the information of the brane box configuration is encoded in the singularity, and viceversa. Since there are aspects of the gauge theory which are easier to analyze in either of both pictures, we hope the dictionary we intend to develop in the present work will also be useful in the understanding of general chiral $N=1$ gauge theories.

The next example we would like to consider is a $k\times k'$ box model, with trivial identifications of the sides of the unit cell, as that shown in figure 8. Thus we start with k NS branes along 012345 and k' NS$'$ branes along 012367. We will consider the case of having an equal number N of D5 branes with world-volume filling 012346. After a T-duality along 4 and 6, the k NS branes are transformed into k KK monopoles, realized as a non-trivial geometry in the coordinates $6'789$. On the other hand the k' NS$'$ branes become KK$'$ monopoles, corresponding to a nontrivial geometry in $4'589$. The resulting space in $4'$,5,$6'$,7,8,9 is a non-compact Calabi-Yau threefold for which we do not have an explicit metric. However, since it arises by the 'superposition' of multi-Taub-NUT metrics, it is easy to uncover the relevant features which control the gauge theory on the D3 brane probes. If we consider the regime where the positions on 789 of the NS branes are close to the origin in the initial configuration, it is clear that the centers of the KK monopoles will lie close to the origin. The space contains a curve of A_{k-1} ALE singularities parametrized by $4'5$ and roughly defined by $x^7=x^8=x^9=0$. Similarly, when the positions of the NS$'$ branes on 589 are close to the origin, the space will contain a curve of $A_{k'-1}$ ALE singularities defined by $x^5=x^8=x^9=0$, and parametrized by $6'7$. Both curves intersect at a point, where the singularity is worse and has the local structure of $C^3/(Z_k\times Z_{k'})$, with the generators θ,ω of Z_k, $Z_{k'}$ acting on $(z_1,z_2,z_3)\in C^3$ as

$$\begin{aligned}
&\theta:(z_1,z_2,z_3)\to(e^{2\pi i/k}z_1,z_2,e^{-2\pi i/k}z_3)\\
&\omega:(z_1,z_2,z_3)\to(z_1,e^{2\pi i/k'}z_2,e^{-2\pi i/k'}z_3)\,.
\end{aligned}\tag{4.4}$$

Here the complex coordinate z_1 corresponds to $x^7,x^{6'}$, the coordinate z_2 refers to $x^5,x^{4'}$, and z_3 to x^8,x^9.

The D3 branes will sit precisely at the singular point, so the structure of the singularity controls the properties of the $\mathcal{N} = 1$ four-dimensional gauge theory. This T-duality argument explains the result we obtained in Section **??** where we observed that the theory on D3 branes on top of such singularity was the same as that obtained in a $k \times k'$ box model.

Finally, and for future convenience, let us notice that through this T-duality map, the box located in the position (i, j) in the $k \times k'$ box grid corresponds to the irreducible representation $\mathcal{R}_{i,j}$ of $Z_k \times Z_{k'}$. This is manifest from our example ii) of section **??**, and will be a useful way of labeling boxes in some arguments.

4.2. Models with non-trivial identifications

The only difficulty in extending the above arguments to a T-duality prescription for a general brane box configuration is the possibility of identifications up to a shift. In the following we show how to handle these cases. Consider a general brane box model, which without loss of generality, we can take to be a $k \times k'$ box model with trivial identification of the vertical sides, and identification of horizontal sides with a shift of p boxes to the left. Our aim is to use T-duality to relate this configuration to some geometry (which admits a local description as C^3/Γ) such that the gauge theory on D3 brane probes reproduces the initial one.

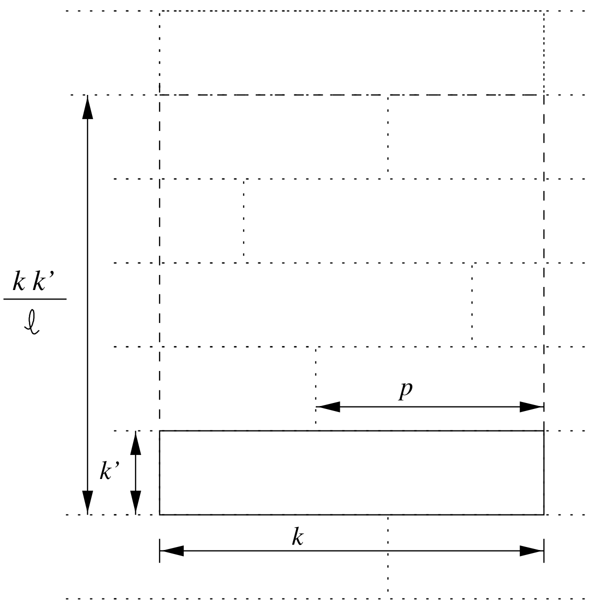

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Figure 9: By adjoining several unit cells, we can define a larger rectangle whose sides have trivial identifications.

In order to avoid the complications coming from the shifted identification, we can adjoin several unit cells until we fill a rectangle for which the identifications of sides are trivial.[3] Figure 9 shows a way of doing this. If we denote by ℓ the greatest common divisor of k and p, $\ell = \gcd(k,p)$, such a rectangle has $k \times kk'/\ell$ boxes. On it, the true unit cell is repeated k/ℓ times. However, it will be useful to consider a 'parent' model where all the $k \times kk'/\ell$ boxes are considered independent. Our original model will be obtained from this one after a $Z_{k/\ell}$ identification, given by a translation in the torus by p boxes to the left and k' boxes upwards.

The strategy we are to follow is first to T-dualize along 46, and then to impose this identification in the resulting T-dual picture. The T-duality of the parent model presents no difficulty since the sides of the rectangle have trivial identifications. The T-dual theory is that of D3 branes sitting at a singularity locally of the form $C^3/(Z_k \times Z_{kk'/\ell})$, with the generators θ, ω of Z_k, $Z_{kk'/\ell}$ acting as

$$\begin{aligned}(z_1, z_2, z_3) &\to (e^{2\pi i/k} z_1, z_2, e^{-2\pi i/k} z_3)\\ (z_1, z_2, z_3) &\to (z_1, e^{2\pi i \frac{\ell}{kk'}} z_2, e^{-2\pi i \frac{\ell}{kk'}} z_3)\,,\end{aligned} \tag{4.5}$$

with z_1, z_2, z_3 defined as in the preceding subsection.

Since our original model had fewer different boxes than the parent model, the final theory should correspond to D3 branes sitting at a less singular point. In order to understand how this can be done, it is illuminating to momentarily consider a similar problem in a $\mathcal{N} = 2$ theory. Consider such theory realized as a $k \times 1$ box model with trivial identifications of sides, and perform a T-duality along 46. This yields, as we know, D3 branes at a C^2/Z_k singularity with generator θ acting as $(z_1, z_3) \to (e^{2\pi i/k} z_1, e^{-2\pi i/k} z_3)$ (forgetting about x^4, x^5 which does not enter the argument in this simpler case). But let us suppose we make a 'mistake' in the choice of the unit cell and consider it to be a $nk \times 1$ box rectangle, without noticing that each box is repeated n times. So, if the nk boxes are considered different, we end up with a T-dual geometry C^2/Z_{nk}, with generator θ' acting as $(z_1, z_3) \to (e^{2\pi i \frac{1}{nk}} z_1, e^{-2\pi i \frac{1}{nk}} z_3)$. The question is how we can correct our 'mistake', the Z_n identification we had missed, once in the dual picture. This is done by noticing that the true T-dual should correspond to C^2/Γ with

[3] This process is the inverse of that in section 3, where starting from a rectangle with trivial identifications we made a choice for the unit cell inside it.

Γ a subgroup of Z_{nk}. In this case we have $\Gamma = Z_{nk}/Z_n \approx Z_k$ with generator $\theta = \theta'^n$. This Z_n action can be viewed as an order n automorphism on the extended Dynkin diagram of A_{nk-1}, a counter-clockwise rotation by n nodes. This action is actually geometrically realized on the two-cycles that resolve the singularity.

A similar discussion applies to our $\mathcal{N} = 1$ case. We had obtained a $C^3/(Z_k \times Z_{kk'/\ell})$ singularity, but the parent model contained an order k/ℓ identification of boxes, which we had not taken into account. The true singularity then must correspond to $\Gamma = (Z_k \times Z_{kk'/\ell})/Z_{k/\ell}$. The $Z_{k/\ell}$ is generated by $\theta^{-p}\omega^{k'}$ (as can be seen by noticing the action relating two identified boxes in the original picture). The representation $\mathbf{3}$ that defines the action of Γ on C^3 is induced from the action of the parent singularity, in the following way. One defines a surjective homomorphism from the set of irreducible representations of the parent singularity to the set of irreducible representations $\mathcal{R}_{\mathbf{I}}$ of Γ, such that $\mathcal{R}_{-p,k'}$ is mapped to the unity $\mathcal{R}_{\mathbf{0}}$. Then Γ acts on C^3 as dictated by the image of $\mathbf{3}$ through this homomorphism. The resulting singularity is independent of the particular homomorphism chosen, as long as it fulfills the mentioned condition.

The meaning of the above procedure is most clear when we recall from previous sections that each box in the brane box configuration corresponds to one irreducible representation of Γ. The fact that some boxes in the $k \times kk'/\ell$ rectangle are identical means that some representations in $Z_k \times Z_{kk'/\ell}$ are to be considered identical. This is accomplished by the homomorphism above, which essentially states that a movement of p boxes to the left and k' upwards takes one box to another copy of the same box.

4.3. Examples

To make the construction somewhat clearer, let us work out a few examples.

i) Let us start considering the 3×1 box model with trivial vertical identifications, and horizontal identifications up to a shift of one box to the left, as shown in figure 6. Thus $k = 3$, $k' = 1$, $p = 1$ and $l = 1$. The model can be understood as coming from a parent 3×3 box model. The T-duality along 46 of such model is very simple, and produces a set of D3 branes on top of a $C^3/(Z_3 \times Z_3)$ singularity, with the generators acting as in (4.5). In order to take into account the Z_3 identifications of boxes to transform the parent model into the true one, we must quotient $Z_3 \times Z_3$ by the Z_3 generated by $\theta^{-1}\omega$. The three equivalence classes in the quotient are $\{1, \theta^2\omega, \theta\omega^2\}$, $\{\theta, \omega, \theta^2\omega^2\}$, and $\{\theta^2, \theta\omega, \omega^2\}$, and so the quotient group is Z_3.

In order to find its action on C^3 we have to relate the irreducible representations of $Z_3 \times Z_3$ with those of Z_3. A homomorphism sending $\mathcal{R}_{2,1}$ to $\mathcal{R}_{\mathbf{0}}$ is

$$\begin{aligned} \{\mathcal{R}_{0,0}, \mathcal{R}_{2,1}, \mathcal{R}_{1,2}\} &\to \mathcal{R}_{\mathbf{0}} \\ \{\mathcal{R}_{1,0}, \mathcal{R}_{0,1}, \mathcal{R}_{2,2}\} &\to \mathcal{R}_{\mathbf{1}} \\ \{\mathcal{R}_{2,0}, \mathcal{R}_{1,1}, \mathcal{R}_{0,2}\} &\to \mathcal{R}_{\mathbf{2}} \end{aligned} \tag{4.6}$$

(the only other choice, with $\mathcal{R}_{\mathbf{1}}$ and $\mathcal{R}_{\mathbf{2}}$ exchanged, leads to identical results). The image of $\mathbf{3} = \mathcal{R}_{1,0} \oplus \mathcal{R}_{0,1} \oplus \mathcal{R}_{-1,-1}$ under this map is $\mathbf{3} = \mathcal{R}_{\mathbf{1}} \oplus \mathcal{R}_{\mathbf{1}} \oplus \mathcal{R}_{\mathbf{1}}$, which defines the action of the final Z_3 on C^3. This completes the construction, showing that the initial brane box model is T-dual to D3 branes at a C^3/Z_3 singularity.

ii) Let us consider a more general case as final example. Consider the brane box model shown in figure 7, which consists of a $k \times 1$ box model with trivial identification of vertical sides, and identifications of horizontal sides accompanied by a shift of p boxes to the left. The parent model is given by a $k \times k/\ell$ box model (where ℓ is the greatest common divisor of k and p, $\ell = \gcd(k,p)$), whose T-dual is a $Z_k \times Z_{k/\ell}$ singularity with generators acting as in (4.5) (in this case $k' = 1$). The order k/ℓ identification is taken into account by computing the quotient by the subgroup generated by $\theta^{-p}\omega$. There are k equivalence classes, the i^{th} of which has the elements $\theta^i(\theta^{-p}\omega)^n$ for $n = 0, \ldots, k/l-1$. The final group is $\Gamma = Z_k$.

Let us find the action on C^3. A natural homomorphism (fulfilling the conditions mentioned above) between the sets of irreducible representations is given by $\mathcal{R}_{i-np,n} \to \mathcal{R}_{\mathbf{i}}$ for $n = 0, \ldots, k/\ell - 1$, and $0 = 1, \ldots, k-1$. Under this map the representation $\mathbf{3}$ $=\mathcal{R}_{1,0}\oplus\mathcal{R}_{0,1}\oplus\mathcal{R}_{-1,-1}$ becomes $\mathcal{R}_{\mathbf{1}}\oplus\mathcal{R}_{\mathbf{p}}\oplus\mathcal{R}_{-\mathbf{p}-\mathbf{1}}$. This defines the action of Z_k on C^3. Notice that the T-duality argument provides in a constructive way the type of singularity that we saw in section 3 reproduces the starting model.

It is time to revisit an open issue we had in our study of realization of field theories using brane box configurations, in section 2. Namely, the fact that different brane configuration can lead to the same field theory, the only difference being that e.g. the horizontal fields in one appear as vertical or diagonal fields in the other. As we mentioned in section 3 all these brane configurations are reproduced by the same singularity simply by changing the correspondence between complex planes in the singularity and horizontal, vertical and diagonal fields in the brane box model. The T duality argument above improves our understanding of the situation. If we start with a singularity C^3/Γ, and wish to relate it to a brane box configuration, we have to perform T duality along the $U(1)$ orbits in two complex planes, say z_1, z_2. More precisely, by this we mean first substituting the singularity by a manifold with the same local behaviour but different asymptotics, so that the mentioned orbits have finite radius at infinity, and the T dual configuration makes sense.[4] The brane configuration that arises will be such that the diagonal fields reproduce the fields $\Phi_{I,I\oplus A_3}$ (associated to the third complex plane z_3, *i.e.* precisely the non T-dualized one). The fields $\Phi_{I,I\oplus A_1}$, $\Phi_{I,I\oplus A_2}$ will map to horizontal and vertical fields. These two latter possibilities are obviously related by the exchange of the roles of NS and NS$'$ branes.

It is then clear that any of the three kinds of fields $\Phi_{I,I\oplus A_i}$, $i = 1,2,3$, can be taken to reproduce the diagonal fields in a T-dual brane box configuration, by merely T-dualizing along the two other directions. This means that T dualities of the same singularity along different directions reproduce the different brane boxes yielding the same four-dimensional field theory. This is a nice result, since it points towards some unifying description of all the brane box models yielding the same field theory.

[4]The situation is analogous to the relation of ALE and Taub-NUT metrics.

4.4. T-duality of wrapped NS fivebranes

In our previous arguments showing the T-duality relation between the brane box configurations and the D3 branes at singularities, the role played by the D branes is quite trivial. We can consider removing them from the picture, and look at the result we have obtained as a T-duality between certain grids of intersecting NS fivebranes wrapping cycles in a torus, and certain non-compact Calabi-Yau threefold geometries. The latter can be roughly described as singularities of the type C^3/Γ with modified asymptotics that make two of the coordinates (x^4 and x^6) compact at infinity.

Such grids have been described as infinite grids on the plane modded out by certain translations, giving rise to identifications of the sides of some unit cell. When the identifications are trivial, the NS fivebranes in the grid wrap cycles of type $(1,0)$ and $(0,1)$ in the torus. When the identifications are non-trivial, the NS fivebranes wrap more complicated cycles in the torus. It is interesting to translate the specific infinite grids we have been studying to the cycles the fivebranes wrap when one effectively restricts to the quotient torus.

One can always define the cycle wrapped by a particular kind of brane, say the NS$'$, to be of type $(1,0)$. This amounts, in the language of infinite grids, to saying that one can always choose a unit cell with trivial identifications of, say, vertical sides. So let us consider the most general such configuration, by now familiar, consisting on a $k\times k'$ box model with trivial identifications of vertical sides and identification of horizontal sides accompanied by a shift of p boxes to the left. The picture corresponding to the following explanations is depicted in figure 10 for a particular example.

The NS$'$ branes wrap a $(1,0)$ cycle which we denote by a. The dual cycle, of type $(0,1)$ is denoted b, and is represented by a slanted line, closed due to the shifted identification. The NS brane wraps a cycle c, represented as a set of vertical lines which

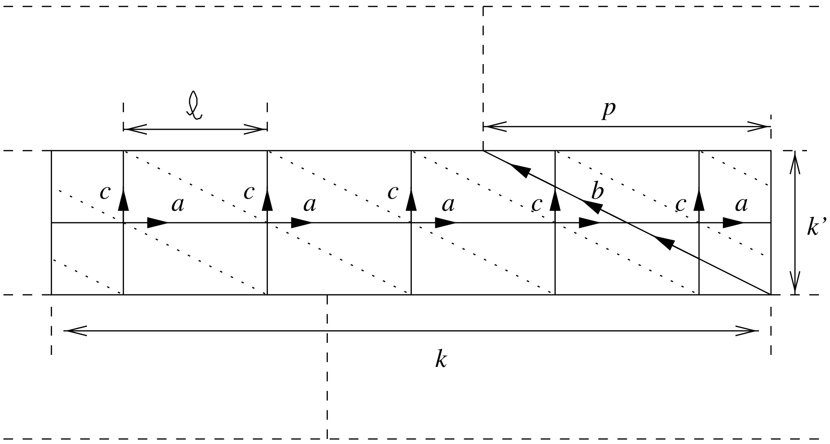

Figure 10: The unit cell in a model with $k=5\ell$, $p=2\ell$. The cycle of the torus wrapped by the NS$'$ branes is the horizontal line labeled 'a'. The cycle 'c' corresponding to the NS branes wraps the vertical direction several times due to the shifted identifications (suggested by the dotted lines). The cycle 'b', corresponding to the slanted line, is the dual to 'a'.

form a closed loop due to the shifted identifications (suggested by dotted lines). This cycle can be expressed in terms of the basic homology cycles, $c = na + mb$. We can determine the type (n, m) of the cycle c that the NS branes are wrapping by simply looking at its intersection number with the basic cycles a and b, $c \cdot a = -m$, $c \cdot b = n$. Recall that the intersection number of two cycles, $c_1 \cdot c_2$, counts the number of their intersection points, with 'plus' signs when the orientation defined by c_1, c_2 (in this order) is positive, and 'minus' signs otherwise. Notice that $a \cdot b = 1$. Due to the shifted identification, a *single* NS brane corresponds to k/ℓ vertical lines in the unit cell, where $\ell = \gcd(k, p)$. Thus we have $c \cdot a = -k/\ell$. Noticing that the vertical lines in the unit cell have an equal spacing of ℓ boxes, we also have $c \cdot b = p/\ell$. Thus, the NS branes wrap cycles of type $(p/\ell, k/\ell)$.

Other choices of the unit cell, for example one with trivial identification of *vertical* sides, yield other labelings of the same cycles, but they are simply related by a $SL(2, Z)$ transformation on the complex structure of the torus.

As a simple example, we can consider the 3×1 box model depicted in figure 6. The NS$'$ brane wraps a $(1, 0)$ cycle, and the NS brane wraps a $(1, 3)$ cycle. A pictorial version of this model is shown in figure 11.

So our considerations in the preceding sections show how to perform T-duality in the following type of configuration: two sets of fivebranes, one spanning 01235 and an arbitrary cycle in a two-torus, another spanning 01237 and another arbitrary cycle in the torus. After T-duality in the two directions of the torus, one obtains a certain manifold which for many purposes can be approximated by a C^3/Γ singularity. The orbifold group Γ is determined from the grid of fivebranes by the recipe presented in subsections 4.1, 4.2, *i.e.* from considerations concerning the four-dimensional theory on D-brane probes of the configuration. This is an amusing aspect of the present study. By looking at two different configurations which lead to four dimensional supersymmetric gauge theories, one is led to these T-dual pairs. The methods presented here actually demonstrate how, starting with one configuration, we can use four dimensional gauge theories to calculate the T-dual partner.

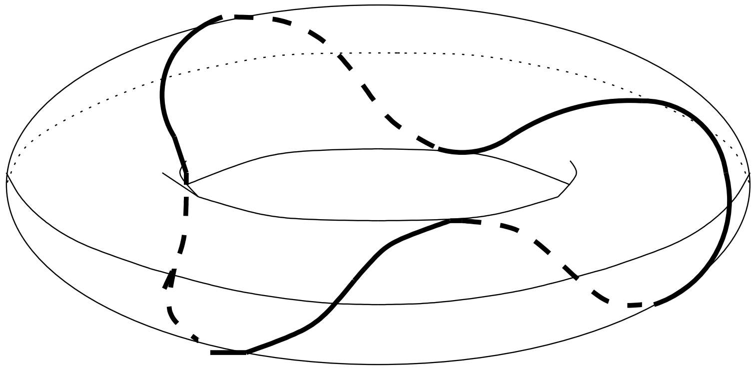

Figure 11: A pictorial representation of a cycle of type $(1, 3)$, which is wrapped by the NS branes in the 3×1 box model of figure 6.

5. Gauge couplings and AdS/CFT correspondence

5.1. The marginal couplings

We have already mentioned in section 2 that the four-dimensional $\mathcal{N} = 1$ gauge theories we are considering have a certain number of marginal couplings; there is a manifold of renormalization group fixed points in the space of couplings. In [3] it was shown that one such marginal parameter existed for each independent horizontal row of boxes in the brane box configuration, another for each independent vertical column of boxes, and another for each independent line of boxes running diagonally from upper right to lower left. Recall that the 'vertical' parameters were interpreted in the brane box model as the independent distances between NS branes. Similarly, the 'horizontal' couplings corresponded to the independent distances between NS$'$ branes. The interpretation of the 'diagonal' parameters is less clear, even though they seem to be related to fields living at the intersection of NS and NS$'$ branes. The overall coupling is determined by the area of the torus parametrized by 4,6.

We would like to achieve some understanding of these field theory parameters in the construction *via* branes at singularities. A quite general family of models where we can study this issue is the field theories obtained as $k \times k'$ box models with trivial identifications of sides. These theories have one overall coupling, $k-1$ 'vertical' marginal couplings, $k'-1$ 'horizontal' couplings, and $r-1$ 'diagonal' ones, where $r = \gcd(k, k')$ as usual.

The singularity that reproduces this field theory in the world-volume of D3 brane probes is $C^3/Z_k \times Z_{k'}$, with the generators θ, ω acting on C^3 as in (2.13).

We would like to identify how these parameters are encoded in the singularity. The overall coupling is given by the string coupling in the usual way. In order to understand the remaining paremeters, it is useful at this point to recall the case of the $\mathcal{N} = 2$ $SU(N)^k$ theories. These models have k marginal couplings, interpreted as one overall and $k-1$ 'vertical' couplings in the brane box construction. These same parameters are interpreted in the T-dual picture of D3 branes at an A_{k-1} singularity as the string coupling, and the $k-1$ integrals a_i, v_i of the NS-NS and RR two-forms over the two-cycles of the resolved ALE space, equations (4.2), (4.3). Since our $\mathcal{N} = 1$ gauge theories have flat directions connecting them to $\mathcal{N} = 2$ theories (by removing either kind of NS fivebranes in the brane box picture), the parameters are expected to be also encoded as integrals of two-forms over two-cycles implicit in the singularity. So we should understand some basic features about the resolution of $C^3/(Z_k \times Z_{k'})$ singularities.

Singular points in the quotient appear from points in C^3 which are left invariant under some element of the discrete group. We can distinguish several types of them. First, there is the (complex) curve defined by $z_1 = z_3 = 0$, and parametrized by z_2, which is invariant under the Z_k subgroup generated by θ. In the quotient it becomes a curve of A_{k-1} ALE singularities. This set of singularities is precisely the only one remaining when in the T-dual brane box model the NS$'$ branes are removed, and one has Higgssed the theory to an $\mathcal{N}=2$ $SU(N)^k$ model. So, it is natural to associate the $k-1$ distances between NS branes (the only parameters that remain in this Higgs branch) to the integrals of the two-form fields over the $k-1$ independent two-cycles that resolve the singularity. Analogously, there is another curve defined by $z_2=z_3=0$, which is left fixed by the $Z_{k'}$ subgroup generated by ω, and which becomes a curve of $A_{k'-1}$ singularities in the quotient. The integrals of the two-form fields over the corresponding two-cycles encode the distances between the NS$'$ branes, since these are the only parameters remaining on the Higgs branch associated to the removal of the NS branes in the brane box picture, which yields a $SU(N)^{k'}$ $\mathcal{N}=2$ field theory.

But there is more to the story. There is yet another curve of singularities. It corresponds to $z_1=z_2=0$, which is left invariant by a Z_r subgroup (with $r=\gcd(k,k')$, as before), generated by $\theta^{k/r}\omega^{-k'/r}$, whose action on C^3 is

$$(z_1,z_2,z_3)\to(e^{2\pi i/r}z_1,e^{-2\pi i/r}z_2,z_3)\,. \tag{5.1}$$

This becomes a curve of A_{r-1} singularities in the quotient. It is easy to see that the field theory has a flat direction connecting it to a $SU(N)^r$ $\mathcal{N}=2$ gauge theory. This breaking, however, is not manifest in the brane box construction[5] and that is the main reason the corresponding $r-1$ parameters were not fully understood in the brane box configuration. In the singularity language, however, the integral of the two-forms over the two-cycles resolving the A_{r-1} singularity are the natural candidates for the remaining $r-1$ parameters. The symmetry between the three kinds of breaking to $\mathcal{N}=2$ in the field theory is manifest in the singularity picture as the symmetry between the three complex planes in C^3. This very nice result provides a geometrical understanding of all the parameters in the gauge theory, and may help in their interpretation in the brane box language.

We should be aware that the resolution of the singularity has not been completed yet. The origin in C^3 is left invariant by all the elements in Γ, and the corresponding singularity in the quotient requires further blow-ups. Consequently, the integrals of p-form fields over the resulting cycles seem to increase the number of parameters in the model. However, there is no contradiction with the above statement that the model contains $k+k'+r-2$ *independent* couplings. The complete space of couplings is certainly larger, but in order to have a conformal theory, so that microscopic couplings exist, the couplings must lie in a $(k+k'+r-2)$-dimensional manifold. It is quite a remarkable fact that these independent parameters are precisely the integrals a_i, v_i of the two-forms over the non-compact divisors (those resolving the curves of singularities, rather than the singularity at the origin). The integrals over the remaining cycles are (possibly complicated) functions of these, and do not provide *new* independent couplings.

[5]However, as in section 2, one can consider another brane box configuration yielding the same field theory, and in which this Higgs branch is manifest.

In the following section we use an argument based on the recent conjecture relating large N gauge theories to string theory on Anti de Sitter spaces to support our identification of the marginal parameters.

Finally, we would like to stress that even though the agreement in the counting of marginal couplings has been shown only for a certain class of models, namely when $\Gamma = Z_k \times Z_{k'}$, the argument also works for other Abelian quotient singularities. Actually, there is a direct relation between closed lines of boxes in the brane box diagram, and subgroups of Γ which leave invariant a complex curve in C^3. It would also be nice to extend these results to more general subgroups of $SU(3)$.

5.2. The AdS/CFT correspondence

In this subsection we connect the discussion of the previous one with the recent conjecture relating the large N limit of gauge theory to string theory on a certain background [25, 26, 27, 28, 29]. The aim of the argument is to find out the number of marginal operators of the four-dimensional $N = 1$ gauge theory. The theories on the world-volume of D3 branes located at orbifold singularities C^3/Γ were proposed in [12, 13] (see also [14, 15, 17, 16]) as simple models to study gauge theories with reduced or with no supersymmetry which were conformal, at least in the large N limit, by using the connection with supergravity/string theory on the space $AdS_5 \times S^5/\Gamma$. The basic requirement for such Γ is that it should act only on S^5, so that the nice property that the group of isometries of the AdS space becomes the conformal symmetry on the boundary (where, roughly, the gauge theory lives) is preserved. Within this class of theories, the detailed correspondence between fields propagating on the AdS_5 and operators on the boundary, analyzed in [26, 29], carries over and can be applied directly. As in the maximally supersymmetric case, the relation between the mass m of p-form field in AdS_5 and the conformal dimension Δ of the operator it couples to in the boundary is

$$(\Delta + p)(\Delta + p - 4) \;=\; m^2 \,. \tag{5.2}$$

One can then hope to be able to compute the conformal dimensions of primary chiral operators in the conformal theory by computing the Kaluza-Klein reduction of ten-dimensional Type IIB supergravity on S^5/Γ to find the masses of fields propagating on AdS_5, in parallel with the comparison made in [26] for the $\mathcal{N} = 4$ case. In [30] this computation was partially performed by taking the KK excitations on S^5 and performing a projection onto Γ-invariant states.

However we must stress that this procedure may not give the complete answer, since it only takes into account the untwisted modes in the quotient. Any possible twisted mode is completely missed by the supergravity approximation, and will be manifest only when the full string theory on the orbifold is considered. This is a very interesting issue, since it will provide indications of how the stringy modes enter the conjectures relating gauge theory and string theory.

Twisted modes appear when S^5/Γ contains singularities, the structure of which is found by looking for fixed points of the action of Γ on S^5. To this end it will be useful to realize the S^5 as the unit five-sphere in an auxiliary C^3 parametrized by (z_1, z_2, z_3)

$$|z_1|^2 + |z_2|^2 + |z_3|^2 \;=\; 1\,. \tag{5.3}$$

The main observation, already made in [12], is that the elements of Γ whose only fixed point is the origin of this C^3, act freely on the S^5, and do not induce singularities in the quotient. The elements in Γ that leave fixed a complex curve in C^3, however, will induce singularities on the quotient S^5/Γ. In our $\Gamma = Z_k \times Z_{k'}$ example the action of Γ on this C^3 is as in (2.13). The curve $z_1 = z_3 = 0$ of fixed points in C^3 intersects the unit five-sphere along the S^1 given by $|z_2|^2 = 1$. This induces a real curve of A_{k-1} singularities in the quotient S^5/Γ. Analogously, there is another S^1, given by $|z_1|^2 = 1$, of $A_{k'-1}$ singularities, and another S^1, $|z_3|^2 = 1$, of A_{r-1} singularities. These real curves are disjoint on the S^5, so there are no further singularities.

Even though the supergravity description is not valid, these singularities are harmless in the full string theory, and there are some states appearing as twisted sectors. The massless twisted fields at each of these Z_n orbifold singularities will be those appearing in Type IIB compactification on A_{n-1} ALE spaces. Namely, there will be $(n-1)$ sets of fields, each containing a two-form and five scalars. The self-dual two-forms appear from the integral of the Type IIB four-form over each of the $n-1$ two-cycles in the resolution of the singularity, two of the scalars from the integrals of the RR and NS-NS two-forms over the two-cycles, and the remaining three scalars from the positions of the corresponding centers in the ALE metric. From the three kinds of singularities, we get $(k-1)+(k'-1)+(r-1)$ sets of such fields. These fields are massless and propagate in $AdS_5 \times S^1$, where S^1 is the corresponding circle of singularities. So one obtains a tower of states propagating on AdS_5, associated to the Kaluza-Klein reduction of these six-dimensional fields on the S^1. It would be interesting to match the masses of these modes with the conformal dimensions of certain operators on the boundary theory. We will do so for the massless scalar modes in AdS_5 operators, leaving the general question for future research.

Let us first discuss the complex scalars coming from the integrals of the B-fields over the collapsed two-cycles. These are massless scalar fields, so from (5.2) we see they must couple to marginal operators in the conformal field theory. We had already counted and identified them. They are given by $\sum_i tr F_i^2$, where the sum runs over the group factors associated to boxes forming independent horizontal, vertical and diagonal lines in the corresponding brane box diagram. Their number is thus $(k-1)+(k'-1)+(r-1)$, precisely the number of massless scalars of the type mentioned. From our analysis of branes at singularities in the previous section, we also infer that the appropriate couplings between the bulk fields and the boundary operators exist, *i.e.* the fields play the role of coupling constants for the gauge theory.

It is thus a fortunate circumstance that there are no further singularities on S^5/Γ. Otherwise the integrals of p-forms over the new cycles would have provided further massless scalar fields propagating on AdS_5. This would require the theory to have more marginal couplings, a fact which is not found in the field theory analysis. The argument above thus provides supporting evidence for our counting and identification of the independent parameters in the gauge theory. Even though the AdS argument is only valid for large N, our identification of the parameters with the integrals of B-fields in section **??** was mainly based on field theory properties valid for all N (namely, Higgs branches breaking to $\mathcal{N} = 2$).

As for the three remaining scalar modes, we see that in the $\mathcal{N} = 2$ case they transform as a triplet of $SU(2)_R$. So they couple to the D-terms of the gauge theory. For the $\mathcal{N} = 1$ theories, these modes couple to whatever operators become the D-terms after the appropriate breaking to $\mathcal{N} = 2$.

We finish this section with some side comments our study of orbifold theories suggests.

The analysis of marginal couplings in $\mathcal{N} = 2$ theories is simple, and can be extended to non-abelian discrete groups Γ as well. In all the quotients S^5/Γ, with Γ and ADE subgroup of $SU(2)$, there will be singularities and twisted sectors. The number of two-cycles in the resolution of the singularity is given by the number of nodes in the corresponding Dynkin diagram. This is also the number of factors in the gauge group, and thus also the number of marginal coupling of the theory (not counting the overall coupling). So again the twisted sector modes are the appropriate fields in AdS to account for certain operators in the gauge theory.

Once these techniques have shown the geometrical features in the AdS picture that underlie the existence of marginal couplings in the gauge theory, we can use such knowledge and apply it even to non-supersymmetric models in the large N limit. It is known [12, 13] that non-supersymmetric theories obtained from D3 branes on top of a R^6/Γ singularity (with Γ a generic subgroup of $SU(4)$) have at least one marginal coupling, which corresponds to the massless dilaton in the AdS. Now we see that if Γ has (real) curves of fixed points on the S^5, yielding ALE singularities in the quotient, the non-supersymmetric theories will have new marginal operators in the large N limit. As an example consider a Z_{10} singularity, with generator θ acting on the R-symmetry quantum numbers of the fermions (in the $\mathbf{4}$ of $SU(4)_R$) through the representation

$$\mathbf{4} = \mathcal{R}_1 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_{-4}. \tag{5.4}$$

The action on the R-symmetry representation $\mathbf{6}$ of the bosons is

$$\mathbf{6} = \mathcal{R}_2 \oplus \mathcal{R}_{-2} \oplus \mathcal{R}_3 \oplus \mathcal{R}_{-3} \oplus \mathcal{R}_3 \oplus \mathcal{R}_{-3}\,. \tag{5.5}$$

The only singularity in S^5/Γ comes from the fixed points of θ^5, and that it is of A_1 type. Thus we expect this non-supersymmetric theory to have *two* marginal couplings. It should not be difficult to construct further examples along this line.

5.3. Strong coupling limits in the gauge theory

One of the interesting points about the identification of parameters we have carried out is that it allows for the comparison of some dynamical field theory phenomena in the brane box and the singularity pictures. As an example, we briefly comment on the limit in which some of the independent parameters in the theory go to zero.[6] It is important to note that the following discussion is valid only to finite gauge theories. For such models, the branes are not bent and the position of the NS branes are good parameters.

Limits with vanishing parameters are obtained in the brane box picture by letting several, say n, NS branes coalesce. This corresponds to setting to zero n of the 'vertical' parameters, and is associated to a strong coupling limit for some of the gauge factors. The most relevant feature of this limit is the appearance of a six-dimensional $U(n)$ gauge symmetry in the world-volume of the NS branes.[7] This is interpreted as an enhanced global symmetry from the point of view of the four-dimensional gauge theory.

One can recover this behaviour in the singularity picture by explicit mapping (via T-duality) of the parameters involved. We have mentioned that the distance between NS branes (and the corresponding Wilson line degrees of freedom) are mapped to the integrals of the Type IIB two-forms over two-cycles implicit in the A_{k-1} singularity. The strong coupling limit we have discussed corresponds to setting these B-fields to zero. In this regime, D3 branes wrapping the two-cycles give rise to tensionless strings. Notice that one of the six dimensions in which this theory lives, x^6 is compact, and T-dualizing along it we recover the picture of gauge symmetry enhancement we had in the brane box construction.

The picture of the strong coupling limit in the singularity language can be translated to the AdS picture without much change, using the information we obtained in section **??**. In such a strong coupling limit, tensionless strings appear propagating on $AdS_5 \times S^1$. The modes propagating on AdS_5 are obtained by mode expansion on the 'internal' S^1. The massless modes in AdS_5 are a multiplet of $U(n)$ gauge bosons, which arise from the tensionless string wrapping the S^1. The gauge symmetry in the bulk is interpreted as a global symmetry on the boundary field theory; the massless fields couple to the corresponding conserved currents on the boundary.

This example illustrates how the T-duality we have established may help in understanding other constructions. Without the intuition provided by the brane box configurations, the enhanced global symmetry observed from the AdS_5 argument would have been harder to interpret. On the other hand, the singularity picture may help in

[6] We are thankful to M. J. Strassler for discussions on the following arguments.

[7] To be precise, in order to get this enhanced symmetry one should also tune the Wilson lines of the world-volume gauge fields along x^4. Thus the enhanced symmetry locus is reached upon tuning n *complex* parameters. There are additional parameters corresponding to 89 positions of the NS branes but they are set to zero in a typical construction.

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understanding some interesting regimes not so intuitive in the brane box picture. For example, those related to setting to zero some diagonal parameters.

6. Non-conformal theories

In this section we explore the singularity picture corresponding to brane box models with different number of D5 branes in each box. The basic ingredients – fractional branes – that enter the definition of the corresponding configurations of branes at singularities have appeared mainly in the context of D0 branes and M(atrix) theory [31, 32, 33], without any reference to configurations of intersecting branes. We will argue these type of objects provide the T-dual of the brane box configurations with non-constant number of D5 branes. Such relation was explored in [7] for the case of $\mathcal{N} = 2$ theories. Other related issues in models with $N = 1$ supersymmetry were discussed in [34].

6.1. Fractional branes

The first relevant observation is that the T duality relation between the grid of five-branes and the singularity does not depend on the distribution of D branes, so the recipe of sections 3 and 4, that relates a given grid to a given singularity (and viceversa), remains valid. Thus, starting with a given brane box configuration we can determine the orbifold group Γ of the singularity picture. We also know how to associate each box with an irreducible representation of Γ. In the following it will be convenient to label the boxes by their corresponding irreducible representation.

The information about the number n_I of D5 branes in the box labeled $\mathcal{R}_I$ is encoded in the singularity picture in how the orbifold group acts on the Chan-Paton indices of the T-dual D3 branes. If n_{tot} denotes the total number of D5 branes in the brane box configuration, $n_{tot} = \sum_I n_I$, the T-dual configuration can be described as an orbifold of C^3 with n_{tot} D3 branes in the covering space. Here the counting includes all the copies under the orbifold action, if present. The action of Γ on the Chan-Paton factors is defined by a n_{tot}-dimensional representation. The adequate choice to reproduce the spectrum in the brane box configuration is

$$\mathcal{R}_{C.P.} = \bigoplus_I n_I \mathcal{R}_I \,, \tag{6.1}$$

as we will show below. Observe that when the number of D5 branes on each box is the same, say N, this representation consists of N copies of the regular representation $\mathcal{R}_\Gamma \equiv \bigoplus_I \mathcal{R}_I$, as should be the case.

The spectrum is determined following the rules in [13]. It is easy to see that it reproduces the spectrum of the field theory obtained in the brane box picture. The gauge group is $\prod_I SU(n_I)$. There are three kinds of chiral multiplets for each I, whose gauge quantum numbers are determined by computing the tensor products $\mathbf{3} \otimes \mathcal{R}_I$.

There are fields, which we denote by $\Phi_{I,I\oplus A_1}$, transforming in the $(\Box,\overline{\Box})$ of $SU(n_I) \times SU(n_{I\oplus A_1})$. Similarly, the fields $\Phi_{I,I\oplus A_2}$ transform in the $(\Box,\overline{\Box})$ of $SU(n_I)\times SU(n_{I\oplus A_2})$, and $\Phi_{I,I\oplus A_3}$ transform in the $(\Box,\overline{\Box})$ of $SU(n_I) \times SU(n_{I\oplus A_3})$. Here it is understood that if some n_I vanishes the corresponding group, and the chiral multiplets charged under it, are not present.

The basic building block of these configurations are, in the brane box picture, models with one D5 branes in one box (say, labeled $\mathcal{R}_I$) and zero in the rest.[8] Correspondingly, there are some basic configurations in the singularity picture, which correspond to a choice of Chan-Paton factors in the representation $\mathcal{R}_{C.P.} = \mathcal{R}_I$ (notice that $n_{tot} = 1$ in these configurations). The D-brane described by these Chan-Paton factors is called a 'fractional brane'. There are different kinds of these objects, each one being characterized by the representation $\mathcal{R}_I$ of its Chan-Paton factors. Their name is due to the observation that a combination of such branes, one for each irreducible representation of Γ, has Chan-Paton factors $\mathcal{R}_{C.P.} = \bigoplus_I \mathcal{R}_I \equiv \mathcal{R}_\Gamma$ and has the interpretation of a (whole) D3-brane in the quotient.

From the rules above, one can determine the world-volume field theory of such configuration.[9] It has no flat directions, and so the branes are stuck at the singular point. This can also be understood by noticing that in the flat cover C^3 of the orbifold we have only one D3 brane, and the only Γ-invariant configurations corresponds to placing it at the origin. This last argument makes it clear that models with several fractional branes may allow for Γ-invariant configurations with branes away from the origin. In the quotient, the corresponding brane will be able to move away from the singularity. The clearest example is having one fractional brane of each kind, $\mathcal{R}_{C.P.} = \bigoplus_I \mathcal{R}_I \equiv \mathcal{R}_\Gamma$, which defines a brane that can move freely in the quotient space C^3/Γ. The world-volume field theory contains the appropriate Higgs branches. Actually, these are clearly visible in the brane box construction. The configuration has one D5 brane in each box, so that they can recombine and leave the grid of NS and NS' along x^5, x^7, x^8, x^9 (additional moduli are provided by the Wilson lines around 4 and 6 of the worldvolume gauge fields). These are the types of objects we have been considering in previous subsections.

In some cases, which will be our main interest in forthcoming considerations, there may be certain combinations of the basic fractional branes which are allowed to move away on a submanifold of C^3/Γ. This type of motion will occur when, in the brane box configurations, we have the same number of D5 branes in each box belonging to *e.g.* a given horizontal row. In such case, the D5 branes in the row can recombine and move away along x^7, stretched between NS' branes. To make the discussion of the singularity

[8]These configurations violate the restrictions on the numbers of D5 branes derived in [4]. Since for the moment we are treating these configurations merely as building blocks, we will ignore this difficulty.

[9]For simplicity we will discuss in the classical limit, where even a single such brane is dynamical, its world-volume $U(1)$ gauge group not being frozen. The discussion extends straightforwardly to other configurations.

picture clearer, we can consider a $k \times k'$ box model with trivial identifications, even though the conclusions hold in other cases as well. The configuration with one D5 brane in the boxes belonging to the j^{th} row is mapped to a set of fractional branes defined by $\mathcal{R}_{C.P.} = \bigoplus_i \mathcal{R}_{i,j}$. The flat direction in the worldvolume field theory implies that this set of fractional branes is allowed to move along the curve of $A_{k'-1}$ singularities in the quotient, but not away from it.

There exists an analogous set of fractional branes defined by $\mathcal{R}_{C.P.} = \bigoplus_j \mathcal{R}_{i,j}$, which is T-dual to a configuration with one D5 brane in the boxes belonging to the i^{th} column, and zero in the others. There is a flat direction in the field theory which allows the D5 branes in the box model to recombine and move away along x^5. This is mapped to moving the set of fractional D3 branes along the curve of A_{k-1} singularities.

Finally, there is a set of fractional branes given by $\mathcal{R}_{C.P.} = \bigoplus_{l=1} \mathcal{R}_{i+l,j+l}$, which is T-dual to a brane box configuration with one D5 brane in all boxes on the diagonal of the box (i, j). Even though it is not obvious in the brane box construction, the field theory contains a flat direction, which corresponds to moving the set of D3 branes along the A_{r-1} curve.

A geometric interpretation of the fractional branes has been proposed in [35, 31, 10], as higher dimensional branes (or bound states thereof) which are wrapping the cycles which are implicit in the singularity of the orbifold. For example, in the case of A_{n-1} ALE singularities, the $(n-1)$ basic kinds of fractional branes (labeled by $\mathcal{R}_i$, for $i = 1, \ldots, n-1$)) can be understood as some sort of D5 branes wrapping the $(n-1)$ independent two-cycles $\Sigma_{i,i+1}$ which resolve the singularity. The fractional brane corresponding to $\mathcal{R}_0$ is associated to the cycle represented by the affine node in the extended Dynkin diagram (this cycle is, homologically, minus the sum of all the rest). The homology relation between the n cycles explains the fact that a set of n fractional branes represents a whole D3 brane in the quotient, which wraps no cycle.

The main reason for this interpretation is the fact that a fractional brane couples to the closed string modulus which controls the blow-up parameters of the corresponding two-cycle. This analysis has been partially extended to the case of C^3/Γ singularities [10], where the fractional branes are understood as D5 and D7 branes wrapping the two- and four-cycles implicit in the singularity. However, as far as we know there is no systematic way of associating a given irreducible representation with a given cycle. It would be interesting to develop such geometrical interpretation, but we will not pursue this issue in the present work. Rather, in the following subsection we will center on a (quite large) family of models for which such geometric interpretation is simple.

6.2. 'Sewing' $\mathcal{N} = 2$ models

The construction of the models we are to consider is as follows. We start with any desired grid of NS and NS' branes, with equal number N of D5 branes. For concreteness we will speak in terms of a $k \times k'$ box model with trivial identifications, but the construction is possible in the general case. In the singularity picture, we have $\Gamma =$

$$\begin{array}{|c|c|c|}\hline N & N & N \\ \hline N & N & N \\ \hline N & N & N \\ \hline\end{array} \; + \; \begin{array}{|c|c|c|}\hline 0 & N_1 & N_2 \\ \hline 0 & N_1 & N_2 \\ \hline 0 & N_1 & N_2 \\ \hline\end{array} \; + \; \begin{array}{|c|c|c|}\hline M_2 & M_2 & M_2 \\ \hline M_1 & M_1 & M_1 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \; + \; \begin{array}{|c|c|c|}\hline L_2 & L_1 & 0 \\ \hline L_1 & 0 & L_2 \\ \hline 0 & L_2 & L_1 \\ \hline\end{array}$$

Figure 12: 'Sewing' $\mathcal{N} = 2$ models: A large family of theories can be obtained by adding together models formed by whole rows, columns and diagonal lines of boxes. Here the numbers denote the number of D5 branes in the box.

$Z_k \times Z_{k'}$, and $\mathcal{R}_{C.P.} = N \bigoplus_{i,j} \mathcal{R}_{i,j}$ The construction proceeds in three steps, which are depicted in figure 12 for a 3×3 case (with trivial identifications).

The first step is to add N_i branes to each of the boxes belonging to the i^{th} column in the grid ($i = 1, \ldots, k-1$).[10] N_i is kept constant within a column, but varies from one column to another. In the singularity picture, we have added some fractional D3 branes, which are described by $\mathcal{R}_{C.P.} = \bigoplus_{i,j} N_i \mathcal{R}_{i,j}$. The worldvolume field theory has flat directions which, in the brane box picture, correspond to moving whole columns of D5 branes along x^5, or in the singularity picture, to moving sets of fractional branes along the curve of A_{k-1} singularities. The configuration of D branes in the singularity is geometrically interpreted as having N_i D5 branes wrapping the i^{th} two-cycle in the resolution of the A_{k-1} singularity, and N D3 branes free to move in the bulk.

The second step is adding M_j D5 branes to the boxes belonging to the j^{th} row. In the singularity picture, the new fractional branes we have added have Chan-Paton factors $\mathcal{R}_{C.P.} = \bigoplus_{i,j} M_j \mathcal{R}_{i,j}$. The geometric interpretation of this set in the singularity picture is having M_j D5 branes wrapping the j^{th} two-cycle in the resolution of the A_{k-1} singularity. These add to the brane we had before. There are two kinds of flat directions, moving either whole rows along x^7, or whole columns along x^5. They are mapped to the independent motions of each kind of fractional brane along the curves of $A_{k'-1}$ and A_{k-1} singularities.

An interesting feature of the models we have obtained after these two steps is that they provide the most general solution to the constraints derived in [4]. These were obtained by considerations on the bending of the NS fivebranes in the brane box model. They state that the numbers $n_{i,j}$ of D5 branes at the box in the position i, j have to fulfill the "sum of diagonals rule", equation (2.4),

$$n_{i,j} + n_{i+1,j+i} = n_{i,j+1} + n_{i+1,j} \,, \tag{6.2}$$

for all i, j. The most general solution to these conditions can be written as

$$n_{i,j} = n_{i,0} + n_{0,j} - n_{0,0} \,. \tag{6.3}$$

[10]Notice that it is redundant to allow for N_0 D5 branes along the 0^{th} column, since one could reabsorb this in a redefinition of N and the N_i's. A similar comment applies in the following steps of the construction.

Equation (6.2) is a simple difference equation and the solution (6.3) is obtained by a double summation over the indices i and j. Defining $N_i = n_{i,0} - n_{0,0}$, $M_j = n_{0,j} - n_{0,0}$, and $N = n_{0,0}$, we can recast (6.3) as

$$n_{i,j} = N + N_i + M_j \,. \tag{6.4}$$

This is precisely the structure of our models, where the number of D5 branes in a box is controlled by the row and column it belongs to.

The claim in [4] is that these are the most general gauge theories that can be realized in the brane box setup. However, notice that in the singularity picture there is a further curve of singularities, around whose two-cycles we can wrap some fractional branes. This is a possibility suggested by the symmetry of the three curves of singularities, and the corresponding models are constructed by the following third step.

The third step is to add L_a D5 branes to each box belonging to a certain diagonal line of boxes, $a = 1, \ldots, r$. In the singularity picture this corresponds to adding D branes with Chan-Paton factors given by $\mathcal{R}_{C.P.} = \bigoplus_{i,j} L_{a(i,j)} \mathcal{R}_{i,j}$, where $a(i,j)$ denotes the label of the diagonal passing through the box in the position (i,j). The geometrical picture is to add L_a D5 branes wrapping the a^{th} two-cycle in the A_{r-1} singularity. The field theory contains some new flat directions, which are mapped to the motion of these fractional branes along the curve of A_{r-1} singularities.

The theories thus constructed satisfy automatically the condition of anomaly cancellation. This can be checked by noticing that at each step in the construction we add vector-like flavours to the gauge factors. However, we would like to point out that the family of models we have just constructed is not the most general one consistent with anomaly cancellation. Consider for example a 3×3 box model with n D5 branes in one box and zero in the others. This anomaly-free configuration does not belong to the class described above.

Nevertheless, we think the family we have constructed is a fairly large class of models, that it includes the most general solution to the constraints in [4], and also that some nice features of the field theories, to be mentioned in what follows, may allow for a study beyond the classical (zero string coupling) approximation.

6.3. The one-loop beta function

One of the simple features of this family of theories is the expression for the one-loop β function of the gauge factors. Let us compute it first from the field theory point of view. Recall the one-loop β function for a $\mathcal{N} = 1$ $SU(N_c)$ theory with N_f (vector-like) flavours is proportional to $b_0 = 3N_c - N_f$.

In the initial configuration, all gauge groups have three flavours, and the one-loop β function vanishes. After the first step, the group in the box (i,j) has increased its rank in N_i units, and its number of flavours increases by $N_{i-1} + N_i + N_{i+1}$, so the b_0 coefficient changes by

$$\Delta_1 b_0 = 2N_i - N_{i-1} - N_{i+1} \,. \tag{6.5}$$

Observe this is the β function of a $\mathcal{N}=2$ $SU(N_i)$ theory with $N_{i-1}+N_{i+1}$ fundamental hypermultiplets. This theory is actually realized along a flat direction of the $\mathcal{N}=1$ theory.

Similarly, after the second step, the group in the box at position (i,j) has increased its number of colours in M_j units, and its number of flavours by $M_j+M_{j-1}+M_{j+1}$. The corresponding change in the one-loop β function is

$$\Delta_2 b_0 = 2M_j - M_{j-1} - M_{j+1}\,. \tag{6.6}$$

Similarly, after the third step, the β function of the group changes by an amount

$$\Delta_3 b_0 = 2L_a - L_{a-1} - L_{a+1}\,, \tag{6.7}$$

where a labels the diagonal line passing through the box (i,j).

The complete beta function is proportional to the sum of the three contributions (6.5), (6.6), (6.7). The "sewing" of the three $\mathcal{N}=2$ theories is quite manifest in the structure of the beta function, and suggests it could also be understood in the brane pictures.

Let us start the discussion in the brane box configurations. After the first step in the construction, the contribution Δ_1 to the one-loop β function can be understood by studying the bending of the NS branes, since the NS$'$ branes do not bend. As in [23] the dependence of the distance between NS branes with some energy scale (in our case, the vev parametrizing the Higgs branch (which is the Coulomb branch in the $\mathcal{N}=2$ theory)) is proportional to Δ_1. We stress that it is actually naive to assume that the dependence of the gauge coupling with the scale is linear, as the fact that there is only one direction in the NS transverse to the D5 branes seems to suggest. The Higgs branch is parametrized by the coordinate x^5, and also by the Wilson lines of the D5 brane world-volume $U(1)$'s along x^4. Thus, the gauge coupling depends on these two coordinates, and actually obeys a two-dimensional Laplace equation, with logarithmic solutions. This, of course, is more intuitive in a T-dual picture where the coordinate corresponding to the Wilson lines is a distance. This is achieved by T dualizing along x^4, and recovering the type IIA configurations of [23].

It is now clear that the bending of the NS$'$ branes takes into account, in a similar way, the contribution Δ_2 to the one-loop β function. The complete answer, as computed from field theory, is given by adding these contributions. For the moment we lack a complete understanding of how this is accomplished in the brane picture, in particular because the two $\mathcal{N}=2$ sub-theories have logarithmic dependence on different Higgs branches. We will assume this to be true, on the basis of simplicity, and symmetry between NS and NS$'$ branes.

Following these lines, it is clear that the third contribution, Δ_3, should be reproduced by some dynamics controlling the diagonal parameters. As we have mentioned, the nature of these is not clear in the brane box picture. Also, the adequate vevs which parametrize the relevant Higgs branch are not manifest. Thus, any improvement on the understanding of the models after step 3 requires some further knowledge about these important issues.

Let us reproduce these results in the singularity picture. After the second step in the construction, we have a set of fractional branes which can move along the curve of A_{k-1} singularities. The coordinates in this curve parametrize the Higgs branch of the $\mathcal{N}=1$ theory, or the Coulomb branch in the corresponding $\mathcal{N}=2$ theory, and provide the appropriate energy scale on which the gauge couplings depend. As we have explained, the gauge coupling for the group arising from the i^{th} column of boxes is encoded in the integral of the Type IIB two-forms over the i^{th} two-cycles implicit in the A_{k-1} singularity. This field varies over the two real dimensional Coulomb branch, and has sources corresponding to the fractional branes wrapped around the cycles intersecting the i^{th} two-cycle. These sources are then the N_{i-1} fractional branes wrapping the $(i-1)^{th}$ two-cycle, the N_{i+1} wrapping the $(i+1)^{th}$, and also the N_i wrapping the i^{th}. They are sources of charge 1, 1 and -2, respectively, as corresponds to the intersection numbers of the cycles. The gauge coupling thus has a logarithmic dependence with the parameter in the two dimensional flat direction, proportional to $2N_i - N_{i-1} - N_{i+1}$.

We can argue in a similar way that after the second step in the construction, the contribution Δ_2 to the one-loop β function is explained by the evolution of the gauge coupling along the Higgs branch parametrized by the positions of the fractional branes on the curve of $A_{k'-1}$ singularities. Finally, since in the singularity picture the diagonal parameters are manifest, one can also understand the contribution Δ_3 that appears in the final theories, after step 3. It appears as the dependence of the gauge coupling with the moduli parametrizing the curve of A_{r-1} singularities. The symmetry among the three types of contributions is once again manifest in the singularity picture, and suggest the complete contribution should be the sum of all three, as found in the field theory computation.

Thus we see that this class of models allows for a nice understanding of the one-loop β function in terms of several ingredients entering the realization using brane box constructions or branes at singularities. One very interesting direction of future research would be to exploit their $\mathcal{N}=2$ structure to extract exact results. It would also be desirable to understand the one-loop β function in other anomaly-free models not belonging to this class.

7. Final comments

In this paper we have studied the T-duality relation between two brane realizations of four dimensional $\mathcal{N}=1$ chiral gauge theories. In the absence of D branes, the map is to be understood as T-duality between certain grids of intersecting NS fivebranes and certain Calabi-Yau threefold geometries, related to C^3/Γ singularities. The D-branes can be interpreted as probes of these configurations. We have shown that the simplest way to argue for this T-duality map is the study and comparison of the four-dimensional $\mathcal{N}=1$ gauge theories that appear in the world-volume of these probes. Using these theories as guideline we have provided systematic recipes to compute the T-dual picture of a given one.

A satisfying result is that the T-duality relates the two known constructions of $\mathcal{N}=1$ finite theories, namely the brane box models and the D3 branes at singularities. These theories have a number of marginal couplings. We have centered our interest in giving them a geometrical interpretation. The T-duality map has proved useful in the understanding the complete set of parameters. 'Diagonal' parameters are not obviously realized in the brane box setup, but appear manifestly in the T-dual singularity picture. Hopefully, this line of thought can lead to their appropriate interpretation in the brane box picture. Another issue where the T-duality has shown its usefulness is in relating the different brane box configurations that give rise to the same field theory.

An interesting point in our research has been the AdS realization of the large N limit of these $\mathcal{N}=1$ theories. We have argued that the marginal operators in the field theory are correctly reproduced by stringy twisted sectors of the S^5/Γ orbifold. An interesting feature of these fields is that they propagate on a six dimensional space $AdS_5 \times S^1$, instead of having a ten-dimensional origin. It is an open question how to treat the Kaluza-Klein tower of states. A possibility is studying Type IIB supergravity on smooth ALE spaces (times a circle) in presence of the RR four-form background. A more practical point of view, along the lines of [36], would be to use the appropriate $\mathcal{N}=4,2$ five-dimensional gauged supergravity.

Finally, we have shown how the T-duality extends to theories which are not conformal. These theories are easily realized in the brane box picture, placing different numbers of D5 branes on each box. We have argued that these configurations map to fractional branes generically stuck at the singularity. We have also shown how to determine the Chan-Paton matrices for these D3 brane, for a given a brane box configuration. An interesting point is that anomaly cancellation in the field theory imposes some restrictions on the possible Chan-Paton matrices. Presumably, the anomaly cancellation follows from some consistency condition on the construction of the orbifold.

We have also presented a quite large family of anomaly-free models, obtained by "sewing" together several $\mathcal{N}=2$ models. A subset of this theories provides the most general solution to the "sum of diagonals" rule, but the complete family is more general, violating that condition in many cases. However the construction in the singularity picture is very symmetric and suggests the consistency of these configurations even at the quantum level.

Even though these theories are $\mathcal{N}=1$ supersymmetric, there are Higgs branches along which $\mathcal{N}=2$ is restored. The theories have a very simple one-loop β function, which we have (partially) explained in terms of the brane pictures.

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